Speedable left-c.e. numbers

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Effective Reals

- We start by reviewing some notions of effective approximability and of randomness for real numbers.

Definition

A **computable approximation** of a real α is a computable sequence $(a_n)_{n\in\omega}$ of dyadic rationals such that $\lim_{n\to\infty} a_n = \alpha$.
A **left-c.e. approximation** of a real α is a strictly increasing computable approximation and a **right-c.e. approximation** of a real α is a strictly decreasing computable approximation.

A real α is a Δ^2_0 **real** if there exists a computable approximation of α.
A real α is **left-c.e.** if there is a left-c.e. approximation of α and is **right-c.e.** if there is a right-c.e. approximation of α.

Notions of randomness

- A **Martin-Löf test** is a uniformly effective sequence of open sets $U_0, U_1, \ldots$, such that the set U_n has uniform measure of at most 2^{-n-1}.
- A **Solovay test** is a uniformly effective sequence of open sets $S_0, S_1, \ldots$, such that the sum of uniform measures of all S_n is finite.

Definition

A sequence A is **Martin-Löf random**, if there is no Martin-Löf test $U_0, U_1, \ldots$, such that $A \in \bigcap_{i \in \omega} U_i$.

- It is well-known that a sequence A is Martin-Löf random if and only if there is no Solovay test $S_0, S_1, \ldots$ such that A is contained in infinitely many of the sets S_n.

Further, we identify any real $\alpha := 0.A$ with its binary representation A.

Speedability

Definition

A left-c.e. (or right-c.e.) real α is **ρ-speedable with respect to its left-c.e. (or right-c.e.) approximation** $(a_n)_{n\to\infty}$ for a constant $\rho \in (0,1)$ if there is a computable strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\liminf_{n\to\infty} \frac{\alpha - a_{f(n)}}{\alpha - a_n} < \rho.$$

Such a function f is called a **speed-up function**.

A left-c.e. or right-c.e. real α is **ρ-speedable** if it is ρ-speedable with respect to some of its left-c.e. (or right-c.e.) approximations.

- When replacing $\liminf$ by $\limsup$ in the definition above, the newly defined concept holds exactly for every computable real α.

Independence of speedability from the constant and approximation

Theorem

A left-c.e. real α that is ρ-speedable for some $\rho \in (0,1)$ is ρ'-speedable with respect to all of its left-c.e. approximations and for all $\rho' \in (0,1)$.

- A similar statement holds for right-c.e. reals.
- The theorem allows us to use the notion of speedability on the field of left-c.e. (or right-c.e.) reals without specifying a left-c.e. (or right-c.e.) approximation and a speeding constant.

Independence from the constant and approximation choice: Proof sketch

The proof follows from three statements:

Lemma

If α is ρ-speedable with respect to some of its left-c.e. approximations $(a_n)_{n\to\infty}$, then α is ρ-speedable with respect to $(a_{f^{[2n]}(0)})_{n\to\infty}$ or $(a_{f^{[2n+1]}(0)})_{n\to\infty}$ via the speed-up function $n \mapsto n+1$.

Lemma

If α is ρ-speedable with respect to some of its left-c.e. approximations $(a_n)_{n\to\infty}$, then α is ρ'-speedable for every $\rho' \in (0,1)$ with respect to the same approximation.

Lemma

If α is ρ-speedable, then α is ρ-speedable with respect to every of its left-c.e. approximations.

Nonspeedability of Martin-Löf random left-c.e. reals

Theorem

Every Martin-Löf random left-c.e. real is nonspeedable.

Proof.

Given a speedable left-c.e. real α, by the proof of the previous theorem we pick some left-c.e. approximation $a_0, a_1, \ldots$ of α that is 1/3-speedable via the successor function $n \mapsto n+1$, so

$$(a_0, a_0 + 3(a_1 - a_0)), (a_1, a_1 + 3(a_2 - a_1)), \ldots$$

is a Solovay test with sum of measures $3(\alpha - a_0)$, which overlaps α infinitely many times. This contradicts the Martin-Löf randomness of α. □

Speedable left-c.e. reals and Solovay degrees

- By the preceding theorem, all left-c.e. reals in the maximum Solovay degree are nonspeedable.
- It can also be shown that all nonhigh left-c.e. reals are speedable.
- Speedability is indeed a degree property for Solovay degrees of left-c.e. reals, i.e., two left-c.e. reals that are both in the same Solovay degree are either both speedable or both nonspeedable.
- It is open whether there are nonspeedable left-c.e. reals outside the maximum Solovay degree, or alternatively, whether there are nonspeedable left-c.e. reals that are not Martin-Löf random.

Speedability of pure Δ^2_0-reals

One can extend the notion of speedability to the general case of Δ^2_0-reals in different ways depending on the meaning of "capturing" α. In this part we introduce the most intuitive generalization which consider the reducing of the distance to α.

Definition

A Δ^2_0 real α is **ρ-speedable with respect to its computable approximation** $(a_n)_{n\to\infty}$ for a constant $\rho \in (0,1)$ if there is a computable strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\liminf_{n\to\infty} \frac{|\alpha - a_{f(n)}|}{|\alpha - a_n|} < \rho.$$

Speedability of pure Δ^2_0-reals

Definition

A Δ^2_0 real is **pure** if it is neither left-c.e. nor right c.e.

Theorem

Every pure Δ^2_0 real is ρ-speedable for every constant $\rho \in (0, 1)$ with respect to each of its computable approximations.

Proof.

Omitted. □