Convergence speed of Cauchy sequences vs relative randomness of their limits

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Outline

Randomness notions for sequences: history

- Statistical approach (von Mises, 1919)
- Information theoretical approach (Solomonoff, 1960; Kolmogorov, 1965; Chaitin, 1966)
- Topological approach (Martin-Löf, 1966)

- **Martin-Löf–Chaitin thesis**:
  The mathematical concept of "Martin-Löf randomness" captures the intuitive notion of an infinite sequence being "random".

Randomness notions for sequences: definitions

Definition

A binary sequence A is **Martin-Löf random**

- $:\iff$ A passes all Martin-Löf tests
- $\iff$ $K(A \restriction n) \geq K(n) - O(1)$, where $K(\cdot)$ means the prefix-free Kolmogorov complexity of the prefix of A of length n
- $\iff$ no computably enumerable martingale has a winning strategy on A

- The restricting to Martin-Löf tests that have a computable measure leads to a weaker notion of randomness called **Schnorr randomness**.

Relative complexity

The prefix-free Kolmogorov complexity induces in a natural way the notions of Kolmogorov ($\leq_K$) and relative Kolmogorov reducibility ($\leq_{rK}$) as measures of relative complexity.

Definition (folklore)

$$A \leq_K B :\iff K(A \upharpoonright n) \leq K(B \upharpoonright n) + O(1)$$

Definition (Downey, Hirschfeldt, LaForte, 2001)

$$A \leq_{rK} B :\iff K\big((A \upharpoonright n)|(B \upharpoonright n)\big) = O(1)$$

The first one can be strengthened to obtain another reducibility $\ll_K$:

Definition (Miyabe, Nies, Stephan, 2016)

$$A \ll_K B :\iff K(B \upharpoonright n) - K(A \upharpoonright n) \underset{n\to\infty}{\rightarrow} \infty$$

From sequences to real numbers

- The field of reals $\mathbb{R}$ is typically defined as completion of $\mathbb{Q}$ relative to the Lebesgue measure λ, i.e., as the set of the equivalence classes of Cauchy sequences (with respect to λ) in $\mathbb{Q}$.
- The last argument motivates to identify a real number α with one of the sequences of rationals $a_0, a_1, \dots \to \alpha$ and to classify the real numbers by the existence of computable Cauchy sequences converging to them (we call the latter ones **computable approximations**) with some special restrictions:

$$\text{CA} = \{\alpha \in \mathbb{R} : \exists \text{ computable } a_0, a_1, \dots \to \alpha\},$$
$$\text{DCE} = \{\alpha \in \mathbb{R} : \exists \text{ comp. } a_0, a_1, \dots \to \alpha : \textstyle\sum_{i=1}^{\infty} |a_{n+1} - a_n| < \infty\}$$
$$\text{LEFT-CE} = \{\alpha \in \mathbb{R} : \exists \text{ comp. } a_0, a_1, \dots \to \alpha : a_0 < a_1 < \dots\}$$
$$\text{COMP} = \{\alpha \in \mathbb{R} : \exists \text{ comp. } a_0, a_1, \dots \to \alpha : |\alpha - a_n| < 2^{-n}\}$$

Relative randomness for real numbers

In the context of randomness, every real number $\alpha = 0.A$ will be identified with its binary representation A.

- Intuition: If a computable approximation $a_0, a_1, \ldots$ of α converges faster than a computable approximation $b_0, b_1, \ldots$ of β, then α is less random than β.

How can the relation "converges faster" be formalized?

Definition (Solovay, 1975; on LEFT-CE: Calude et al., 1998)

A left-c.e. α is Solovay reducible to a left-c.e. β (written $\alpha \leq_S \beta$) if for every left-c.e. approximation $b_0 < b_1 < \ldots$ of β there exists a left-c.e. approximation $a_0 < a_1 < \ldots$ of α such that $\frac{\alpha - a_n}{\beta - b_n}$ has an upper bound.

- The notion of Solovay reducibility relates to the left-c.e. reals α and β without fixed connection with a concrete pair of left-c.e. approximations $a_n \nearrow \alpha$ and $b_n \nearrow \beta$.

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How can the relation "converges faster" be formalized?

Definition (Solovay, 1975)

A real α is **Solovay reducible** to a real β if there exist a constant c and a partial translation function $g : \mathbb{Q} \to \mathbb{Q}$ such that $0 < \frac{\alpha - g(q)\downarrow}{\beta - q} < c$ for every $q < \beta$.

- The original definition of Solovay reducibility uses the notion of translation function that maps rationals on $(-\infty, \beta)$ in rationals on $(-\infty, \alpha)$. Both definitions coincide on LEFT-CE.

Relative randomness for real numbers on LEFT-CE

- (Solovay, 1975) $\alpha \leq_S \beta \implies \alpha \leq_{rK} \beta \implies \alpha \leq_K \beta$.
- (Kučera, Slaman, 2001) The Martin-Löf random reals form the highest $\leq_S$-degree on LEFT-CE.
- (Miyabe, Nies, Stephan, 2016) $\alpha \ll_K \beta \implies \alpha \leq_S \beta$.

Relative randomness for real numbers and the speed of convergence on LEFT-CE

Idea: Given two left-c.e. reals α and β, instead of just requiring the existence of an upper bound for $\frac{\alpha - a_n}{\beta - b_n}$ for two fixed left-c.e. approximations $a_n \nearrow \alpha$ and $b_n \nearrow \beta$, we require the existence of a limit point $\lim \frac{\alpha - a_n}{\beta - b_n} \in \mathbb{R}$.

Theorem (Barmpalias, Lewis-Pye, 2017)

Let α and β be left-c.e. reals such that β is Martin-Löf random. Then $\exists d = \lim \frac{\alpha - a_n}{\beta - b_n}$, which does not depend on the choice of the computable approximations $a_0 < a_1 < \dots$ and $b_0 < b_1 < \dots$ of α and β, respectively.
Moreover, $d = 0$ if and only if α is not Martin-Löf random.

For $\alpha = \beta$ and arbitrary left-c.e. approximations, this refinement yields the notion of speedability.

Randomness for real numbers and speedability on LEFT-CE

Definition (Merkle, Titov, 2020)

A left-c.e. real α is **nonspeedable** if for every two left-c.e. approximations $a_0 < a_1 < \dots$ and $a'_0 < a'_1 < \dots$ of α it holds that $\lim \frac{\alpha - a'_n}{\alpha - a_n} = 1$.

- (Merkle, Titov, 2020; Barmpalias, Lewis-Pye, 2017 (implicitly)) Every Martin-Löf random left-c.e. real is nonspeedable.
- (Hölzl, Janicki, 2024) There exists a Martin-Löf nonrandom left-c.e. real, which is nonspeedable.

Relative randomness for real numbers and the speed of convergence on LEFT-CE

Definition

For two left-c.e. reals α and β, we write $\alpha \ll_S \beta$ if for every left-c.e. approximation $b_n \nearrow \alpha$ there exists a left-c.e. approximation $a_n \nearrow \beta$ such that $\lim \frac{\alpha - a_n}{\beta - b_n} = 0$.

- For $\alpha \in \mathrm{COMP}$ and $\beta \notin \mathrm{COMP}$, it holds that $\alpha \ll_S \beta$.
- (Barmpalias, Lewis-Pye, 2017) $\alpha \ll_S \Omega \iff \alpha$ is not ML random.
- (Titov, 2024) $\alpha \ll_K \beta \implies \alpha \ll_S \beta$.

Relative Schnorr randomness on LEFT-CE

Definition (Downey, Griffiths, 2004)

A real α is **Schnorr reducible** to a real β, written $\alpha \leq_{Sch} \beta$, if for every computable measure machine M there is a computable measure machine N such that $K_N(A \upharpoonright n) \leq K_M(A \upharpoonright n) + O(1)$.

Definition (Merkle, Titov, 2022)

A real α is **total Solovay reducible** to a real β, written $\alpha \leq_S^{tot}$ if $\alpha \leq_S \beta$ via a total translation function.

- (Downey, Griffiths, 2004) Schnorr random reals are closed upwards relative to $\leq_{Sch}$.
- (Merkle, Titov, 2022) On LEFT-CE: $\alpha \leq_S^{tot} \beta \implies \alpha \leq_{Sch} \beta$.

Relative randomness: CA case

How can the notion **converges faster** be generalized on non-monotone computable approximations?

Extensions of Solovay reducibility outside of LEFT-CE

There are several attempts to adapt the concept of the Solovay reducibility on the non-left-c.e. numbers, besides the original approach of Solovay. All of them coincide on LEFT-CE.

Definition (Zheng, Rettinger, 2004)

A c.a. real α is **2a-Solovay reducible** to a c.a. real β (written $\alpha \leq_S^{2a} \beta$) if there exist computable approximations $a_0, a_1, \dots \to \alpha$ and $b_0, b_1, \dots \to \beta$ such that $\frac{|\alpha - a_n|}{|\beta - b_n| + 2^{-n}}$ has an upper bound.

Definition (Titov, 2023; Barmpalias, Lewis-Pye, 2017 (implicitly))

A real α is **monotone Solovay reducible** to a real β (written $\alpha \leq_S^{m} \beta$) if there exist a constant c and a monotone partial translation function $g : \mathbb{Q} \to \mathbb{Q}$ such that $0 < \frac{\alpha - g(q)\downarrow}{\beta - q} < c$ for every $q < \beta$.

Extensions of Solovay reducibility on DCE: 2a-Solovay reducibility

- (Zheng, Rettinger, 2004) DCE reals are closed upwards relative to $\leq_S^{2a}$.
- (Zheng, Rettinger, 2004) $\alpha \leq_S^{2a} \beta \implies \alpha \leq_{rK} \beta \implies \alpha \leq_K \beta$.
- (Rettinger, Zheng, 2005) Martin-Löf random reals form the highest $\leq_S^{2a}$-degree on DCE.
- (Titov, 2023) On DCE: $\alpha \leq_S^m \beta \implies \alpha \leq_S \beta \implies \alpha \leq_S^{2a} \beta$.

Relative randomness for real numbers and the speed of convergence on DCE

Analogous to the left-c.e. case, we can consider the specific case of $2a$-Solovay reducibility of a d.c.e. real α to a d.c.e. real β with the limit point $d = \lim \frac{|\alpha - a_n|}{|\beta - b_n|}$ for some d.c.e. approximations $a_n \to \alpha$ and $b_n \to \beta$.

Just as in the left-c.e. case, we obtain for $d = 1$, $\alpha = \beta$ and independence of the choice of d.c.e. approximations the notion of $2a$-nonspeedability and for $d = 0$ a strengthened reducibility $\alpha \ll_S^{2a} \beta$.

Proposition (Titov, 2024; based on Miller, 2017)

Let α and β be d.c.e. reals such that β is Martin-Löf random. Then $\exists d = \lim \frac{|\alpha - a_n|}{|\beta - b_n|}$, which does not depend on the choice of DCE approximations $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ of α and β, respectively. Moreover, $d = 0$ if and only if α is not Martin-Löf random.

Randomness for real numbers and speedability on DCE

Definition

A d.c.e. real α is **DCE-nonspeedable** if for every two d.c.e. approximations $a_0, a_1, \ldots$ and $a'_0, a'_1, \ldots$ of α it holds that $\lim \frac{|\alpha - a'_n|}{|\alpha - a_n|} = 1$.

- (Barmpalias, Fang, Merkle, Titov, 2024) on DCE:

$$\text{DCE-nonspeedability} \overset{\text{sic!}}{\iff} \text{Martin-Löf randomness}$$

Relative randomness for real numbers and the speed of convergence on DCE

Definition

For two d.c.e. reals α and β, we write $\alpha \ll_S^{2a} \beta$ if for every d.c.e. approximation $b_n \to \beta$ there exists a d.c.e approximation $a_n \to \alpha$ such that $\lim \frac{|\alpha - a_n|}{|\beta - b_n|} = 0$.

- For $\alpha \in$COMP and $\beta \notin$COMP, it holds that $\alpha \ll_S^{2a} \beta$.
- (Titov, 2024; based on Miller, 2017)

$$\alpha \ll_S^{2a} \Omega \iff \alpha \text{ is not Martin-Löf random.}$$

Open question: $\alpha \ll_K \beta \overset{?}{\Longrightarrow} \alpha \ll_S^{2a} \beta$.

Extensions of Solovay reducibility on CA

On CA, the 2a-Solovay reducibility does not inherit all of the closure properties that it fulfills on DCE.

- For every two c.a. reals α and β, there exist computable approximations $a_n \to \alpha$ and $b_n \to \beta$ such that $\limsup \frac{|\alpha - a_n|}{|\beta - b_n|} = \infty$.

Hence, the Theorem of Barmpalias and Lewis-Pye cannot be generalized from DCE on CA using the same reducibility notion.

- For every c.a. real α, there exist two computable approximations $a_n \to \alpha$ and $a'_n \to \alpha$ such that $\lim \frac{|\alpha - a'_n|}{|\alpha - a_n|} = 0$.

The second fact implies that the 2a-Solovay reducibility does not produce any reasonable notions of CA-speedability.

Extensions of Solovay reducibility on DCE: monotone Solovay-reducibility

The latter two deficiencies of $\leq_S^{2a}$ can be fixed by using the stronger monotone version of Solovay reducibility.

Theorem (Titov, 2023)

- *For a real α and a Martin-Löf random real β, such that $\alpha \leq_S^m \beta$ via some translation function g, there exist a constant $d = \lim\limits_{q \nearrow \beta} \frac{\alpha - g(q)}{\beta - q}$, which does not depend on the choice of the translation function g.*
  Moreover, $d = 0$ if and only if α is not Martin-Löf random.
- *Every Martin-Löf random real is monotone nonspeedable.*
- LEFT-CE *and* RIGHT-CE *are closed downwards in $\mathbb{R}$ relative to $\leq_S^m$.*

The latter fact implies that, in general, $\alpha \ll_K \beta$ does not imply $\alpha \leq_S^m \beta$ on CA.

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