Monotone Solovay Reducibility

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Outline

1. Background
2. Previous results
3. Main result

Martin-Löf randomness

- We start by reviewing some notions of randomness, effective approximability, Solovay reducibility and speedability.
- A **Martin-Löf test** is a uniformly effective sequence of open sets $U_0, U_1, \ldots$, such that the set U_n has uniform measure of at most 2^{-n-1}.
- A **Solovay test** is a uniformly effective sequence of open sets $S_0, S_1, \ldots$, such that the sum of uniform measures of all S_n is finite.

Definition

A sequence A is **Martin-Löf random**, if there is no Martin-Löf test $U_0, U_1, \ldots$, such that $A \in \bigcap_{i \in \omega} U_i$.

- It is well-known that a sequence A is Martin-Löf random if and only if there is no Solovay test $S_0, S_1, \ldots$ such that A is contained in infinitely many of the sets S_n.

Further, we identify any real $\alpha := 0.A$ with its binary representation A.

Effective reals

Definition

A **computable approximation** of a real α is a computable sequence $(a_n)_{n\in\omega}$ of dyadic rationals such that $\lim_{n\to\infty} a_n = \alpha$.
A **d.c.e. approximation** of a real α is a computable approximation $(a_n)_{n\in\omega}$ of α that fulfills $\sum_{i\in\omega} |a_{n+1} - a_n| < \infty$.
A **left-c.e. approximation** of a real α is a strictly increasing computable approximation.

A real α is a Δ^2_0 **real** if there exists a computable approximation of α
A real α is **left-c.e.** (**d.c.e.**) if there is a left-c.e. (d.c.e.) approximation of α.
The set of all left-c.e. reals will be denoted as **LEFT-CE**.

Solovay reducibility and its monotone version

Definition

A real α is **(monotone) Solovay reducible** to a real β, written $\alpha \leq_S \beta$ ($\alpha \leq_S^m \beta$), if there is a constant $c > 0$ and a (monotone nondecreasing) partial computable function $g\colon \mathbb{Q} \to \mathbb{Q}$ such that for all $q < \beta$ it holds that $g(q)\downarrow < \alpha$ and $\frac{\alpha - g(a_n)}{\beta - b_n} < c$.

The classical version has been introduced by Solovay in 1975, the monotone version implicitely by Merkle and Titov in 2021. On LEFT-CE, these two notions coincide.

The motivation of requiring the monotony of the translation function outside of LEFT-CE is the idea of considering the notion of speedability (see later) as a monotone Solovay self-reducibility, that coincides on LEFT-CE with the original speedability notion (Merkle, Titov, 2021).

A further equivalent characterization of $\leq_S$ on LEFT-CE:

Equivalent definition (left-c.e. case)

*A left-c.e. real α is **Solovay reducible** to a real β, written $\alpha \leq_S \beta$, if there is a constant $c > 0$ and, for every pair of left-c.e. approximations $(a_n)_{n\in\omega}$ of α and $(b_n)_{n\in\omega}$ of β, respectively, there is a computable index function $f\colon \mathbb{Q} \to \mathbb{Q}$ such that, for all $n \in \omega$, it holds that $\frac{\alpha - a_{f(n)}}{\beta - b_n} < c$.*

Monotone Solovay reducibility: Properties

- $\leq_S^m \implies \leq_S$ ($\implies \leq_S^{1a}$).
- on LEFT-CE: $\leq_S^m \iff \leq_S$ ($\iff \leq_S^{1a}$).
- The classes Δ_0^2, DCE, LEFT-CE, RIGHT-CE, COMP are closed downwards relative to $\leq_S^m$.
- Martin-Löf random reals are closed upwards relative to $\leq_S$ (and therefore, for $\leq_S^m$).
- on LEFT-CE: Martin-Löf random left-c.e. reals form the highest degree with respect to $\leq_S$.

Speedability

Idea: we call a number speedable if it is monotone Solovay reducible to itself with a constant 1 and a function g fulfilling some additional property.

Definition

A real α is **speedable** if $\alpha \leq_S^m \alpha$ via the constant 1 and a monotone function g and there exists a sequence $q_n \nearrow \alpha$ of dyadic rationals converging to α such that

$$\frac{\alpha - g(q_n)}{\alpha - q_n} \leq \rho.$$

Again, on LEFT-CE, these two notions coincide.
An equivalent characterization of speedability on LEFT-CE:

Equivalent definition (left-c.e. case)

*A left-c.e. real α is **speedable** if there exists a left-c.e. approximation $(a_n)_{n\in\omega}$ of α, a computable index function $f\colon \mathbb{Q} \to \mathbb{Q}$ fulfilling $f(n) \geq n$ for every n and a constant $\rho < 1$ such that it holds that*

$$\frac{\alpha - a_{f(n)}}{\alpha - a_n} < \rho$$

for infinitely many n.

Previous results: On LEFT-CE

Theorem (Barmpalias, Lewis-Pye, 2017, rational form)

If α and β are left-c.e. and β is ML random, then, for any monotone function g, such that $\alpha \leq_{\mathrm{S}}^{m} \beta$ via g, and any left-c.e. approximation $(b_n)_{n\in\omega}$ of β, there exists

$$\lim_{n\to\infty} \frac{\alpha - g(b_n)}{\beta - b_n}$$

that does not depend on the choice of the function g witnessing $\alpha \leq_{\mathrm{S}}^{m} \beta$ nor the sequence $(b_n)_{n\in\omega}$.

Previous results: On LEFT-CE

On LEFT-CE, $\leq_S^m$ coincides with $\leq_S$ that allows an equivalent characterization via an index translation function, hence the theorem can be reformulated as in the original version:

Theorem (Barmpalias, Lewis-Pye, 2017 index form)

If α and β are left-c.e. and β ML random, then, for every two left-c.e. approximations $(a_n)_{n\in\omega}$ and $(b_n)_{n\in\omega}$ of α and β, respectively, there exists

$$\lim_{n\to\infty} \frac{\alpha - a_n}{\beta - b_n},$$

which does not depend on the choice of the sequences $(a_n)_{n\in\omega}$ and $(b_n)_{n\in\omega}$.

For $\alpha = \beta$, it implies the nonspeedability of Martin-Löf random left-c.e. reals:

Corollary (Explicitly: Merkle, Titov, 2020)

Every ML random real left-c.e. real is nonspeedable.

Conjectures

Conjecture (Titov, Merkle, 2021)

If $\alpha \leq_{\mathrm{S}}^{m} \beta$ *via a function* g *and* β *ML random, then, for every monotone sequence* $q_n \nearrow \beta$*, there exists*

$$\lim \frac{\alpha - g(q_n)}{\beta - q_n},$$

which does not depend on the choice of g *and* $(q_n)_{n\in\omega}$*.*

Does the Barmpalias-Lewis-Pye Theorem hold true outside of left-c.e. reals?

Conjecture (Titov, Merkle, 2020)

Every left-c.e. real is Martin-Löf random if and only if it is nonspeedable.

Is speedability the characteristic property of Martin Löf random left-c.e. reals?

Conjectures

Conjecture (Titov, Merkle, 2021)

If $\alpha \leq_S^m \beta$ via a function g and β ML random, then, for every monotone sequence $q_n \nearrow \beta$, there exists

$$\lim \frac{\alpha - g(q_n)}{\beta - q_n},$$

which does not depend on the choice of g and $(q_n)_{n\in\omega}$.

Does the Barmpalias-Lewis-Pye Theorem hold true outside of LEFT-CE? JA (main result)

Conjecture (Titov, Merkle, 2020)

Every left-c.e. real is Martin-Löf random if and only if it is nonspeedable.

Is speedability the characteristic property of ML random left-c.e. reals? NEIN, → next talks

Main result

Theorem

If $\alpha \leq^m_S \beta$ *via a function* g *and* β *ML random, then, for every monotone sequence* $q_n \nearrow \beta$*, there exists*

$$\lim \frac{\alpha - g(q_n)}{\beta - q_n},$$

which does not depend on the choice of g *and* $(q_n)_{n\in\omega}$*.*

- In contrast to the left-c.e. case, a sequence $q_n \nearrow \beta$ does not need to be computable.
- For $\alpha = \beta$, the theorem straightforwardly implies the monotone nonspeedability of ML random reals, that has been explicitly proven by Merkle and Titov in 2021.
- Outside of left-c.e. reals, the converse does not hold true: a pure d.c.e. (and therefore, ML nonrandom) real, which cannot be reducible to itself via any nonconstant monotone computable function, can be constructed using finite injury method (side result).

Left-c.e. case: construction idea (Miller, 2018)

Assuming the existence of two rational constants $c < d$ such that

$$(\exists^\infty n : \frac{\alpha - g(q_n)}{\beta - q_n} < c) \wedge (\exists^\infty n : \frac{\alpha - g(q_n)}{\beta - q_n} > d),$$

we consider for a given left-c.e. approximation $q_n \nearrow \beta$ an alternating sequence of local minima of the function

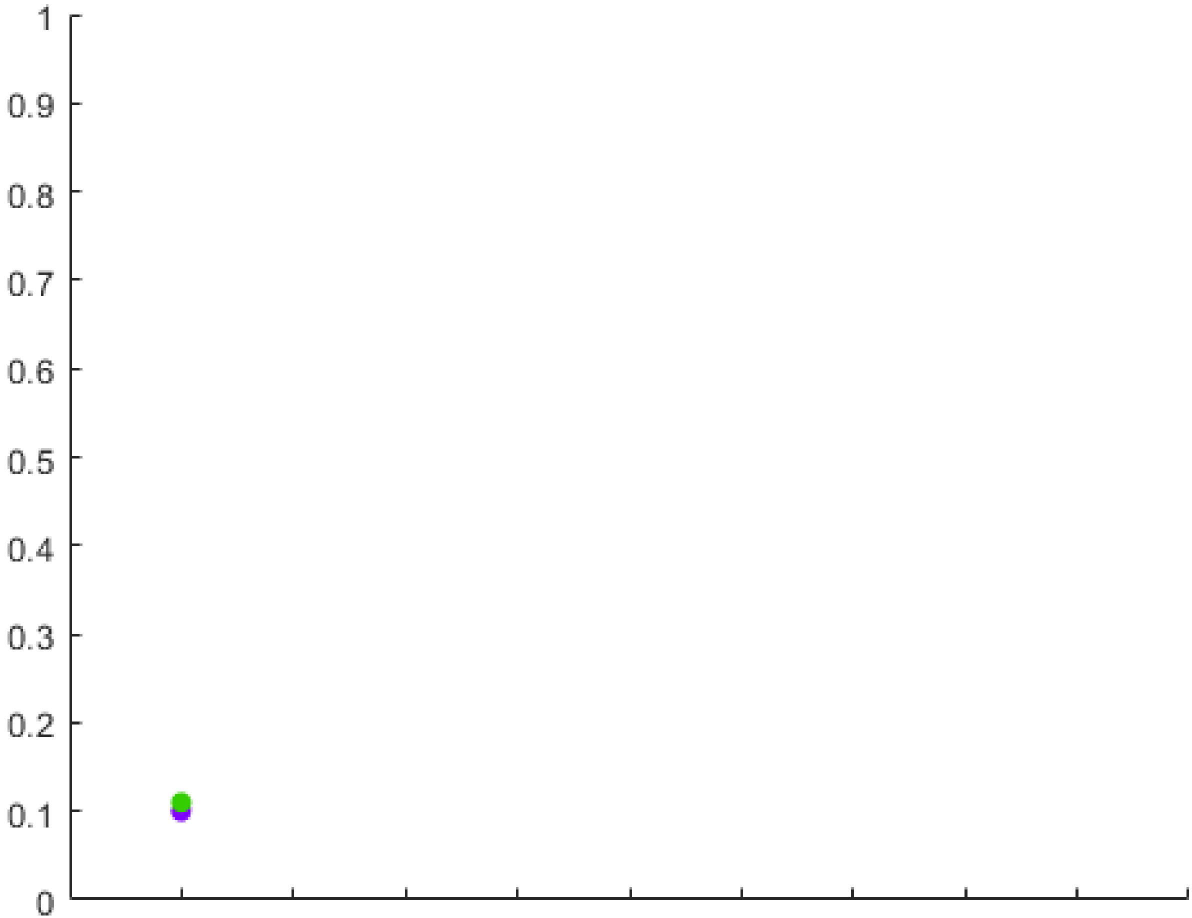
$$\delta(q_n) = g(q_n) - dq_n \left(< (\alpha - d\beta) \text{ in case } \frac{\alpha - g(q_n)}{\beta - q_n} < d \right)$$

and local maxima of the function

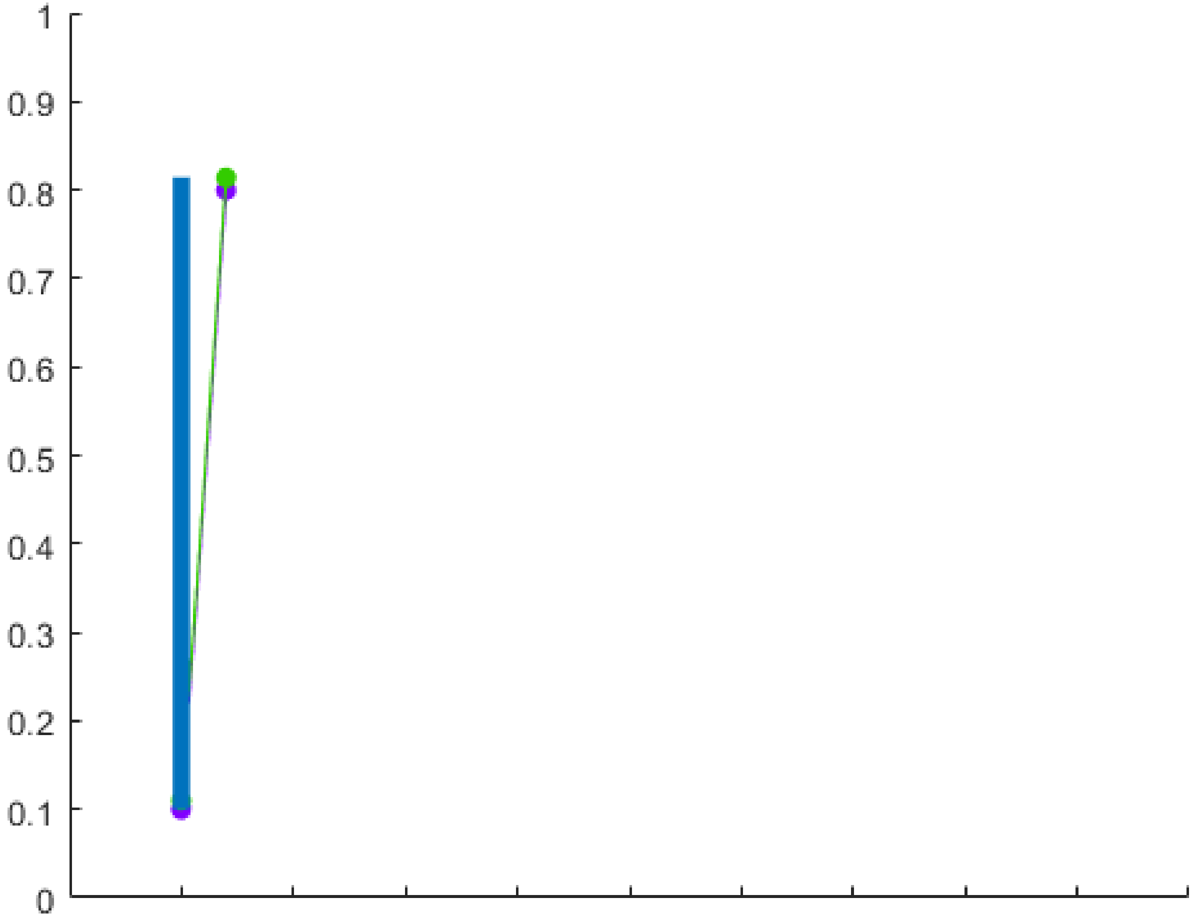
$$\gamma(q_n) = g(q_n) - cq_n \left(> (\alpha - c\beta) \text{ in case } \frac{\alpha - g(q_n)}{\beta - q_n} > c \right)$$

to construct a Solovay test covering $(d - c)\beta$ infinitely many times, that leads us to contradiction with the ML randomness of β.

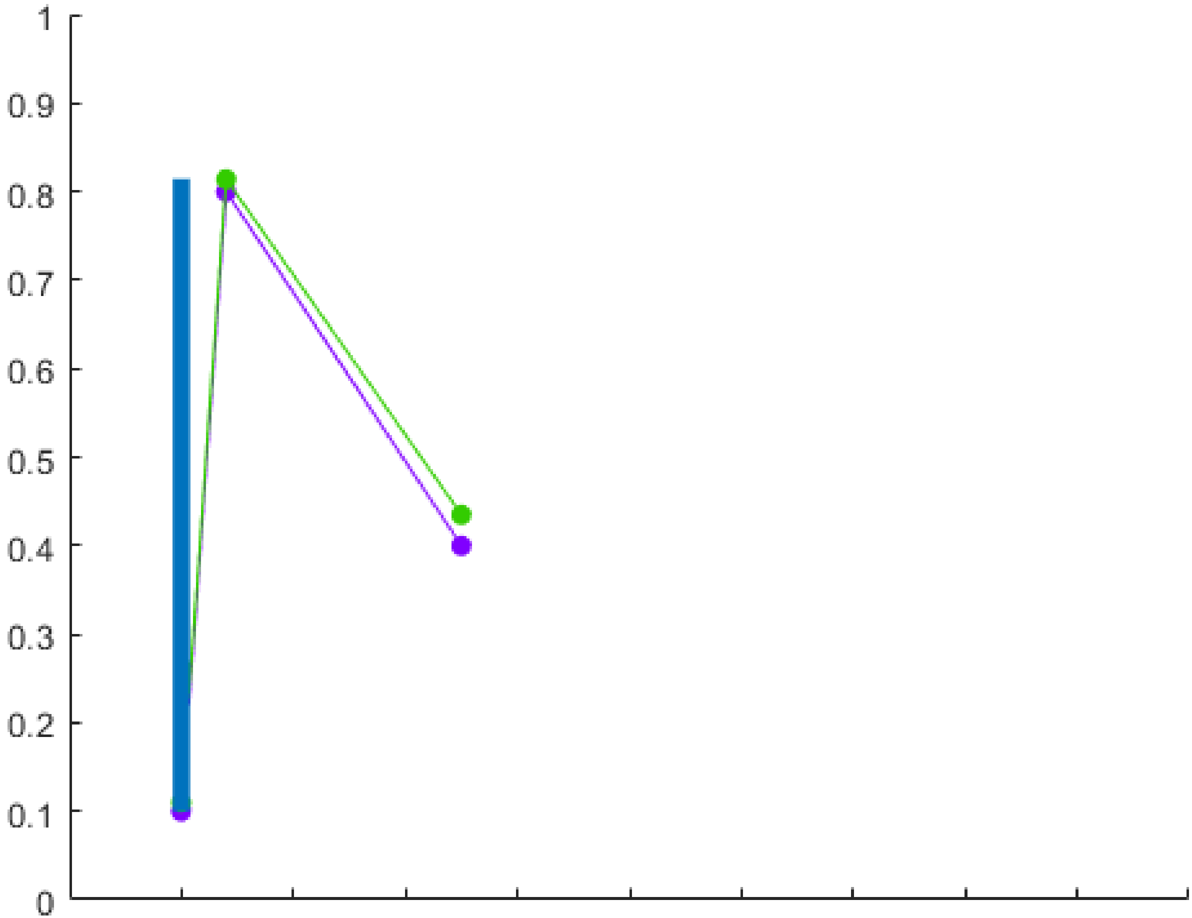
Test construction: first example



Test construction: first example



Test construction: first example



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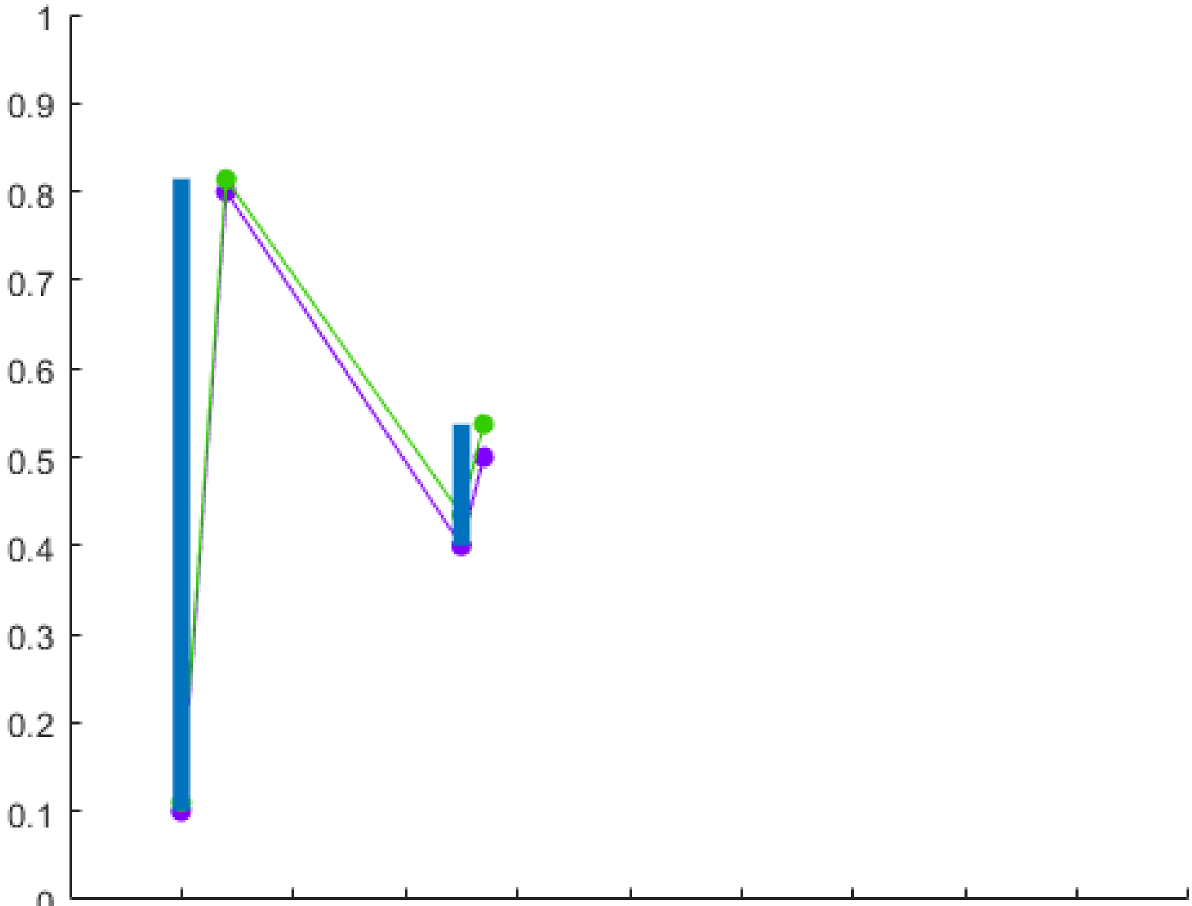

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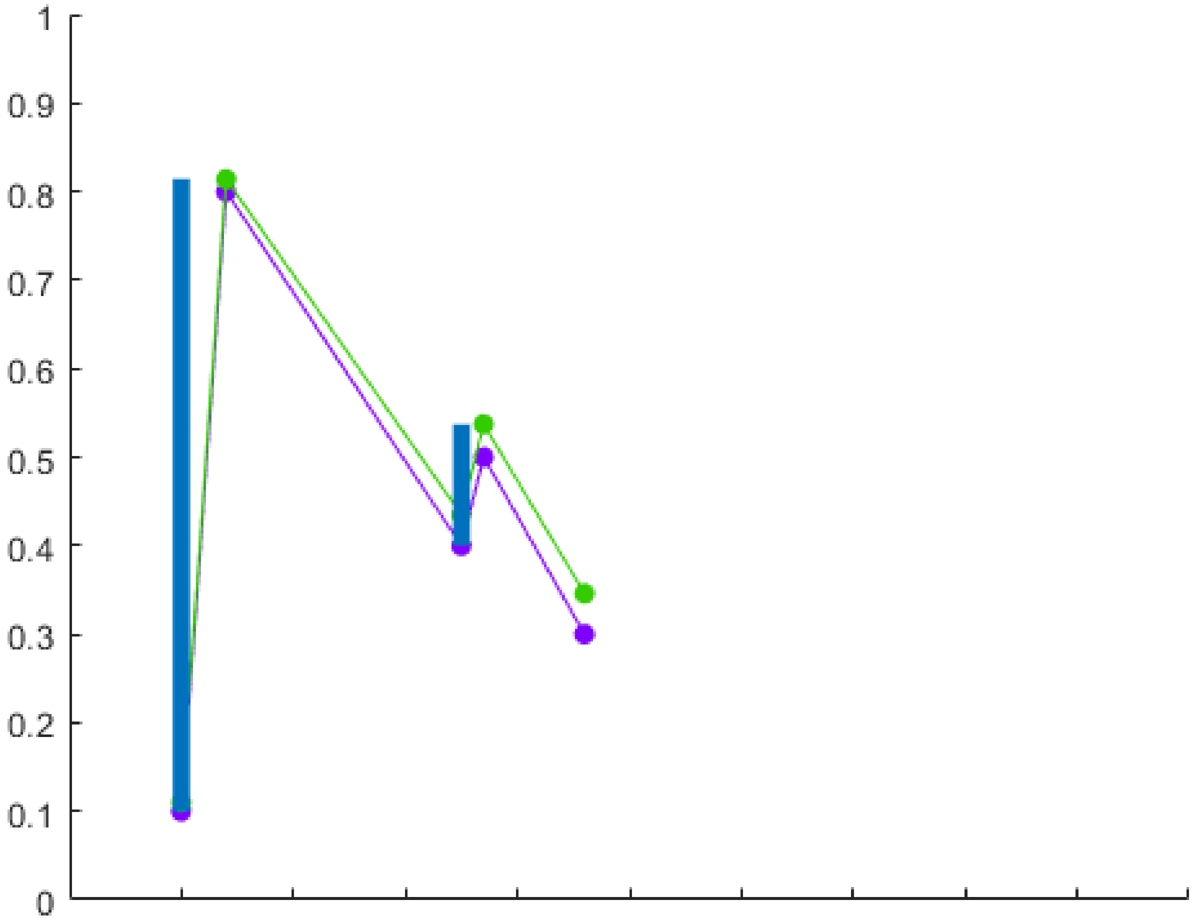

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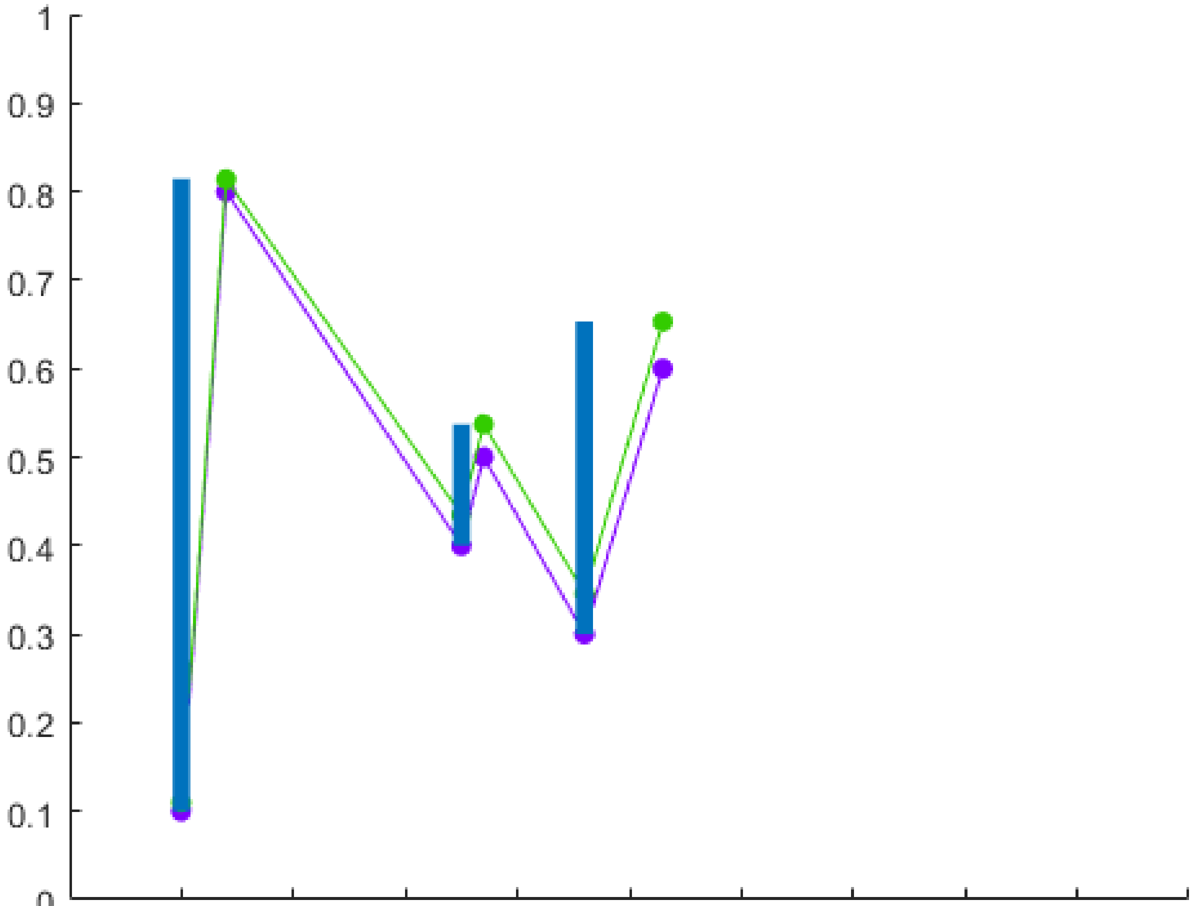
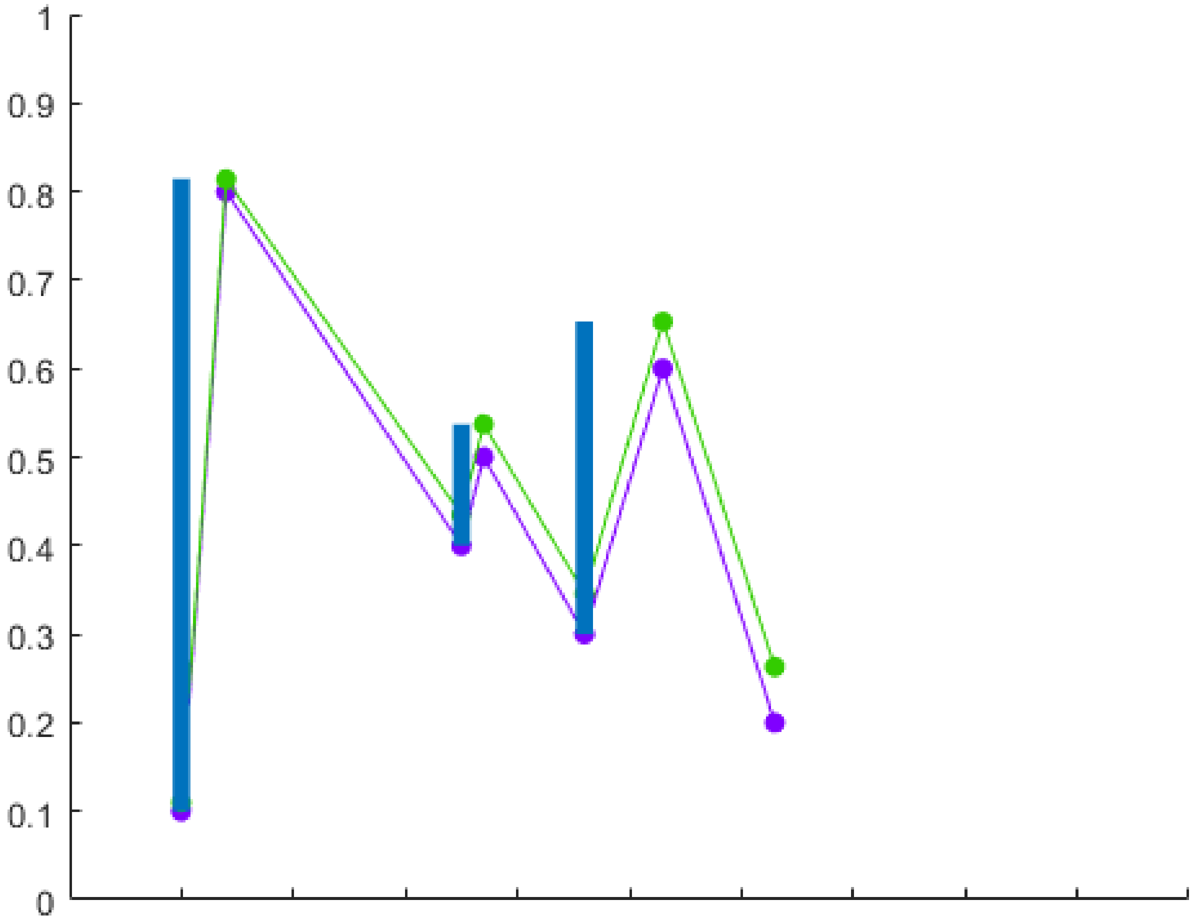
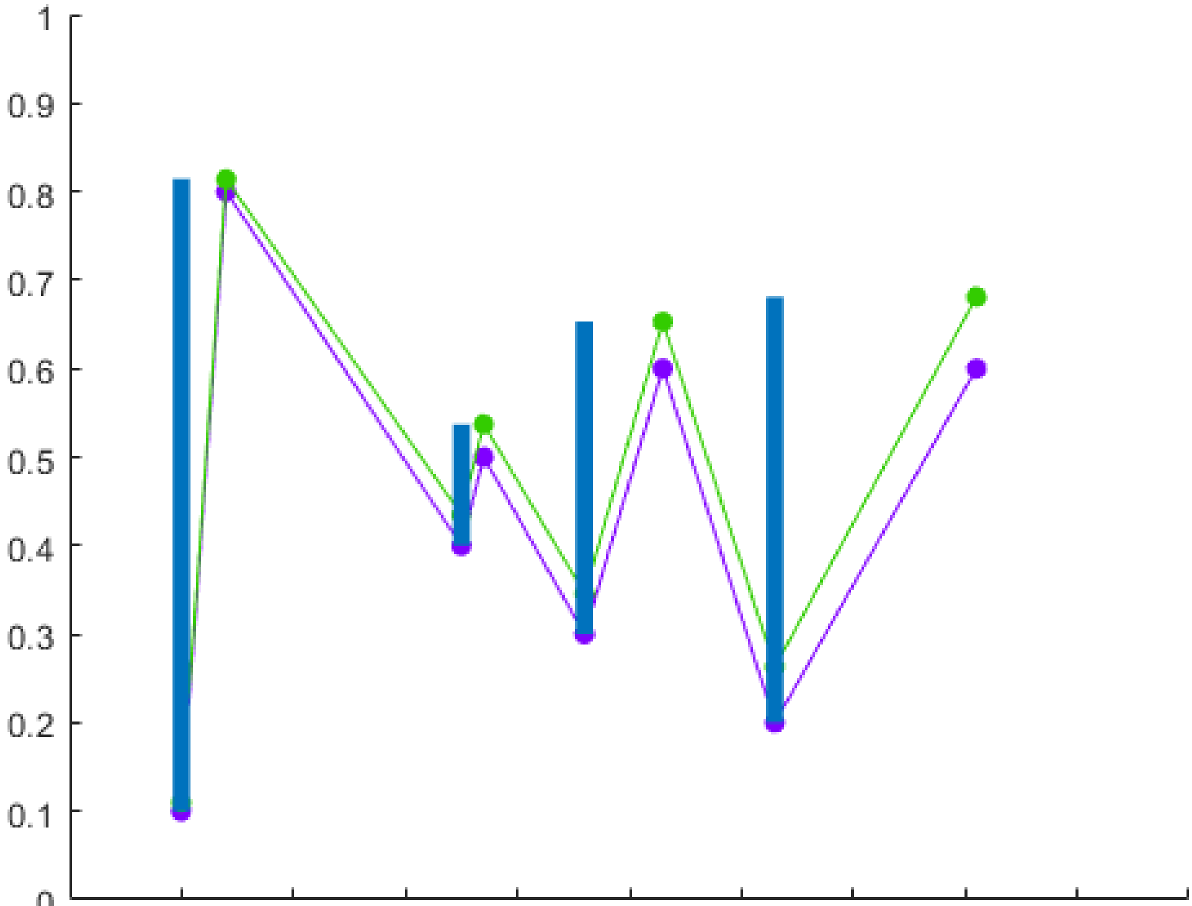


Test construction: first example



Test construction: first example



Left-c.e. case: properties of the constructed test

- if $q_{s_0}, q_{t_0}, q_{s_1}, q_{t_1}, \dots$ is an alternating sequence of points such that, for every i, q_{s_i} is a δ-minimum on $(q_{t_{i-1}}, \beta)$ and q_{t_i} is a γ-minimum on (q_{s_i}, β), then, after enumerating of q_{t_n}, the real $(d-c)\beta$ will be covered by interval $[\gamma(q_{s_i}) - \delta(q_{s_i}), \gamma(q_{s_i}) - \delta(q_{s_i})]$ for every $i \leq n$, hence $(d-c)\beta$ will be covered by the Solovay test infinitely many times.
- after every enumerated point q_n, the measure increase of the currently constructed finite test is bounded from above by XXX, hence, the measure of the whole Solovay test is bounded by XXX

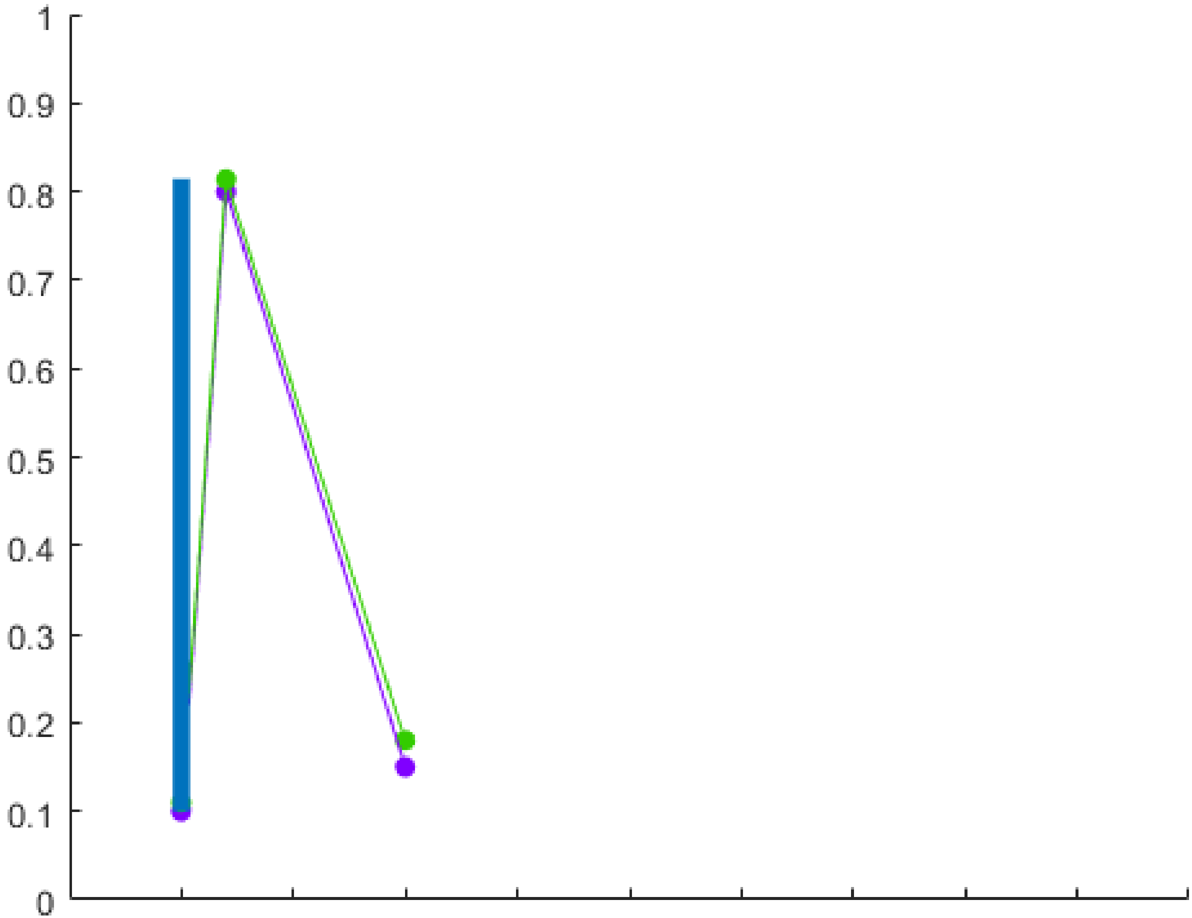
Left-c.e. case $\to$ general case

In nonleft-c.e. case, we do not approximate β from below, but enumerate the domain of Solovay translation function g.

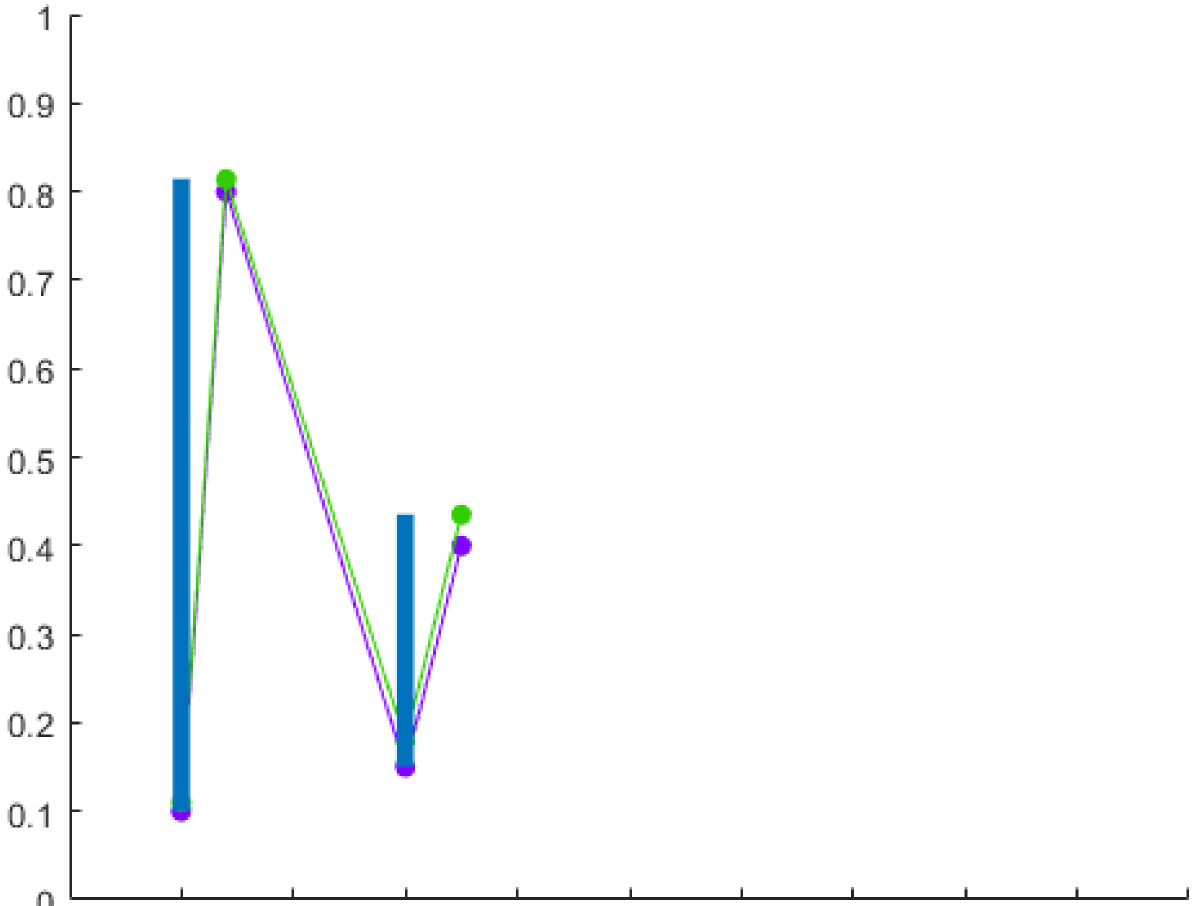
- The new points q_n do not appear in an increasing order, although, while emunerating of $dom(g)$, the amount of points in the alternating sequence of δ-minima below $\alpha - d\beta$ and γ-maxima above $\alpha - c\beta$ still goes to infinity on $[0, \beta)$.
- Hence, the intuitive idea to fit the test construction to the general case is to repeat, after enumerating every new point q_n in $dom(g)$, the finite test construction for the set $\{q_0, \ldots, q_n\}$ and extend the finite test constructed for the set $\{q_0, \ldots, q_{n-1}\}$ at the previous stage upto the new one.
  So, the infinite covering of $(d - c)\beta$ by the constructed infinite test as well as finiteness of the its measure will be inherited from the same properties of the original test construction for left-c.e. case.

In what follows, we see that this idea generally does not work.

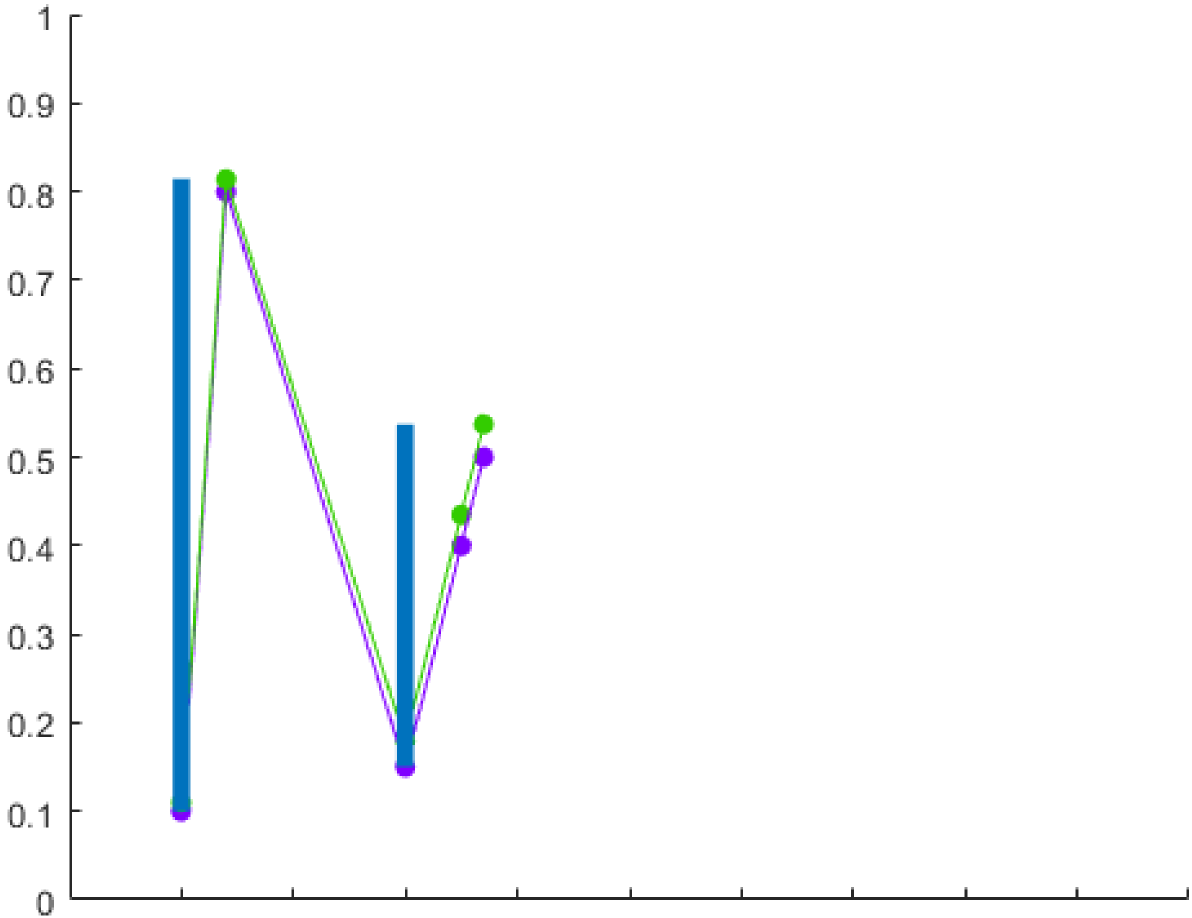
Test construction: second example



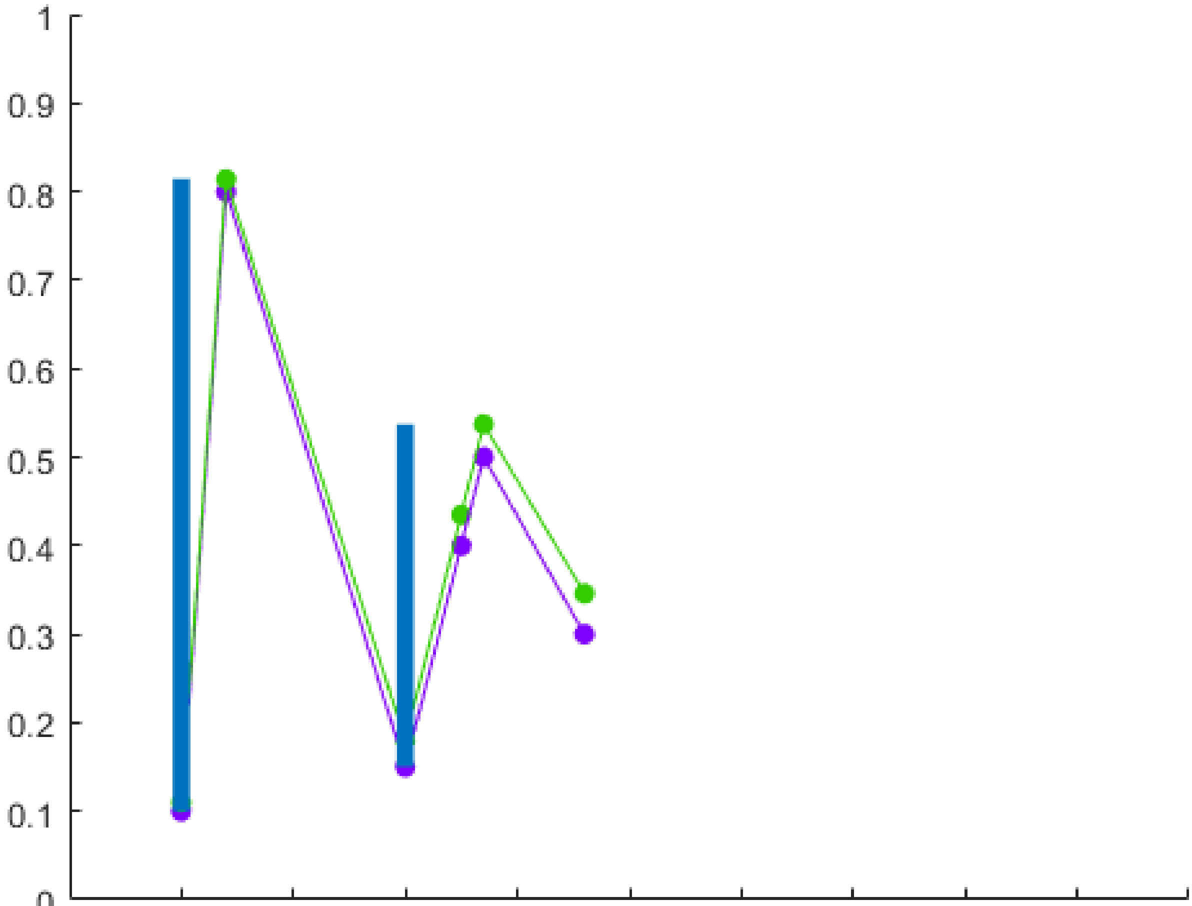
Test construction: example



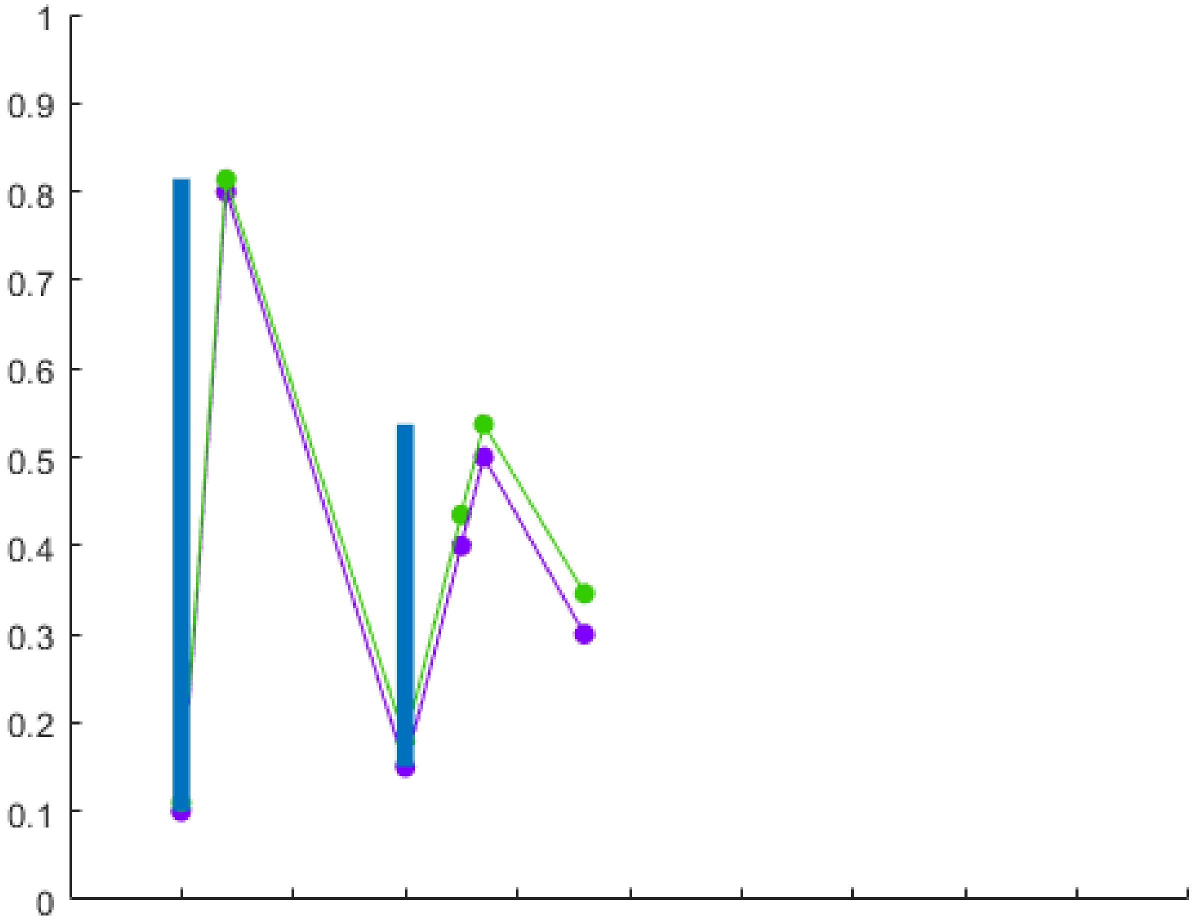
Test construction: example



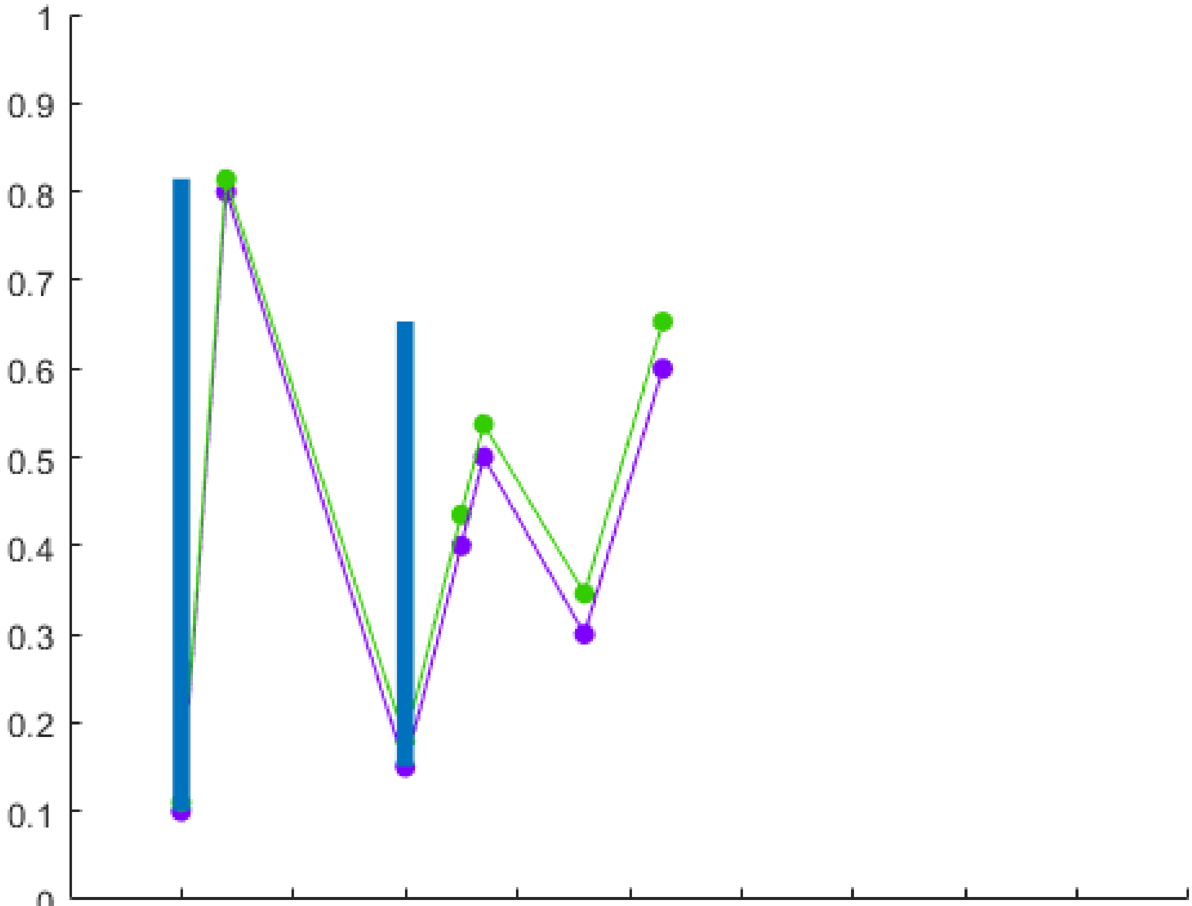
Test construction: example



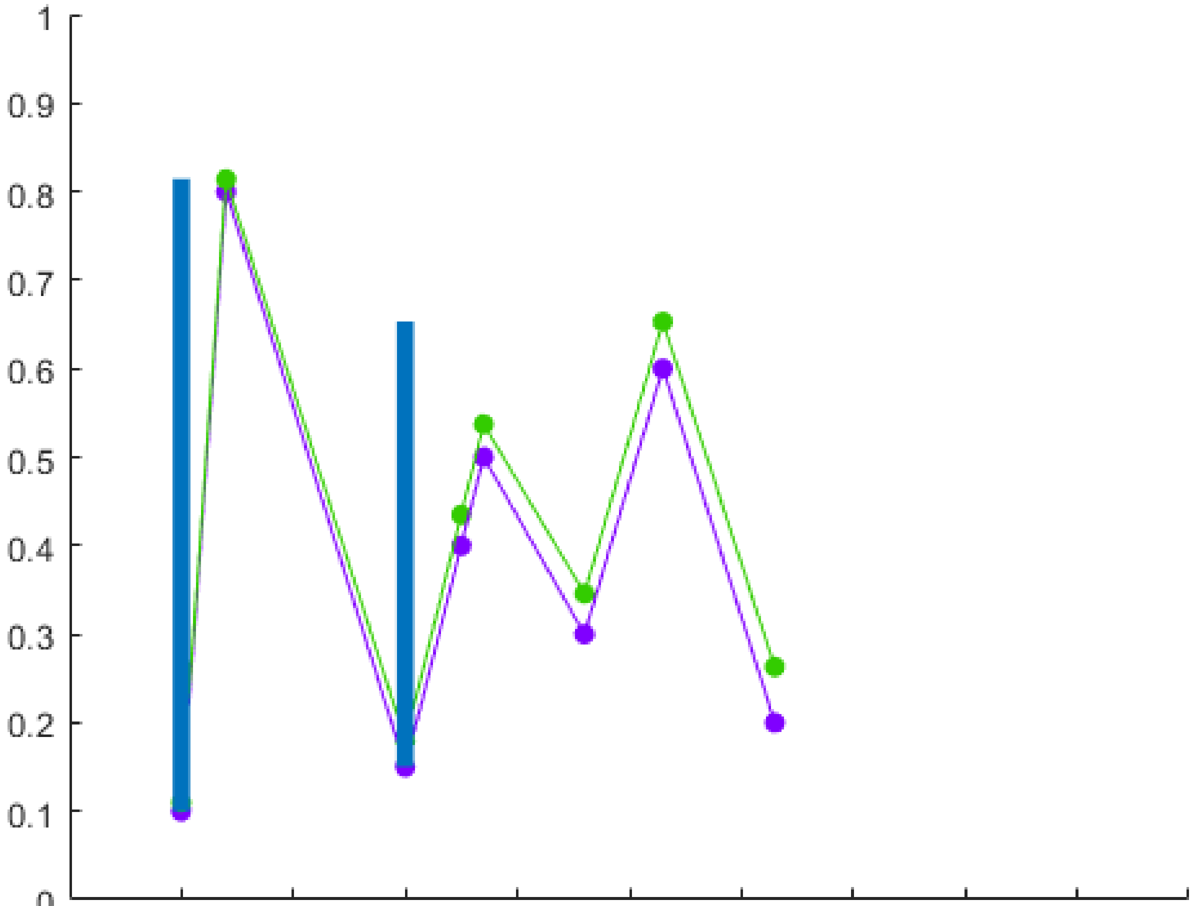
Test construction: example



Test construction: example



Test construction: example



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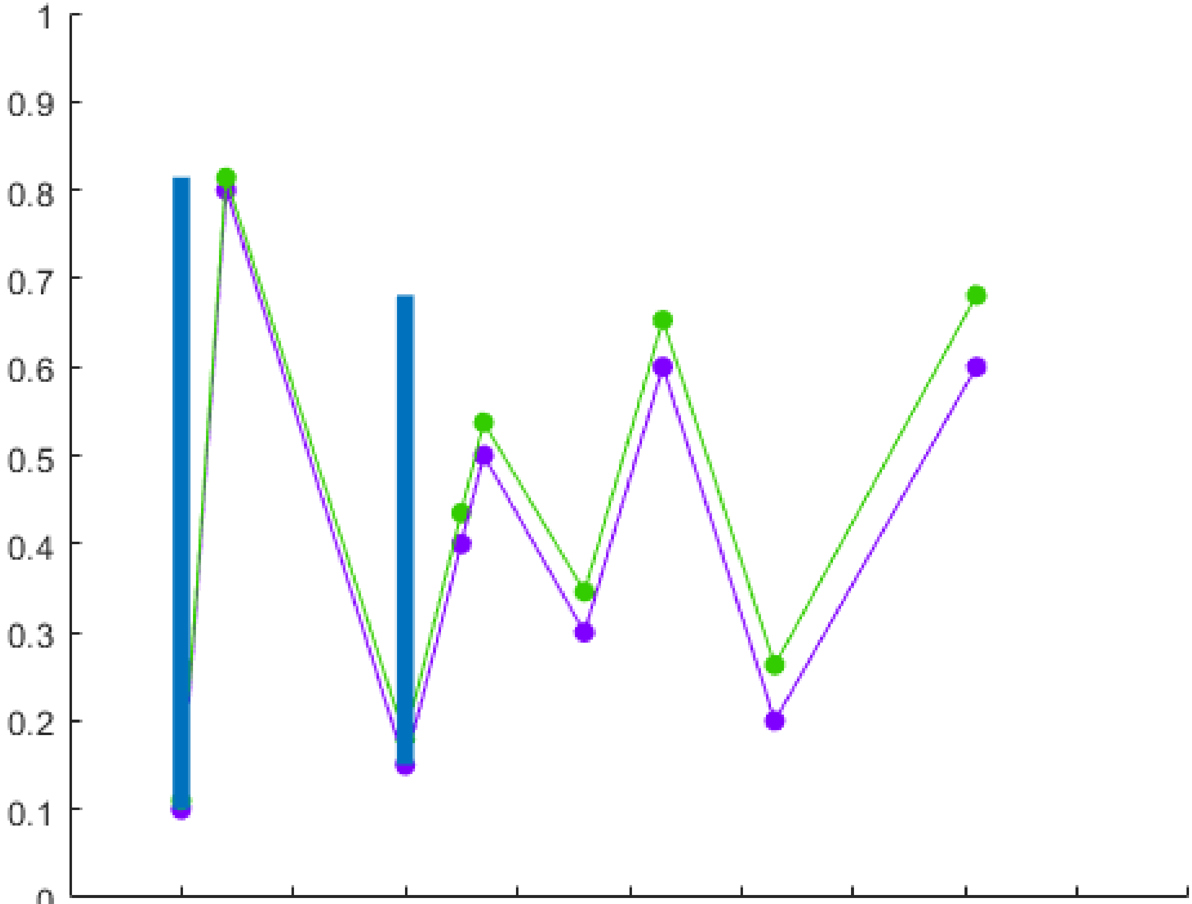
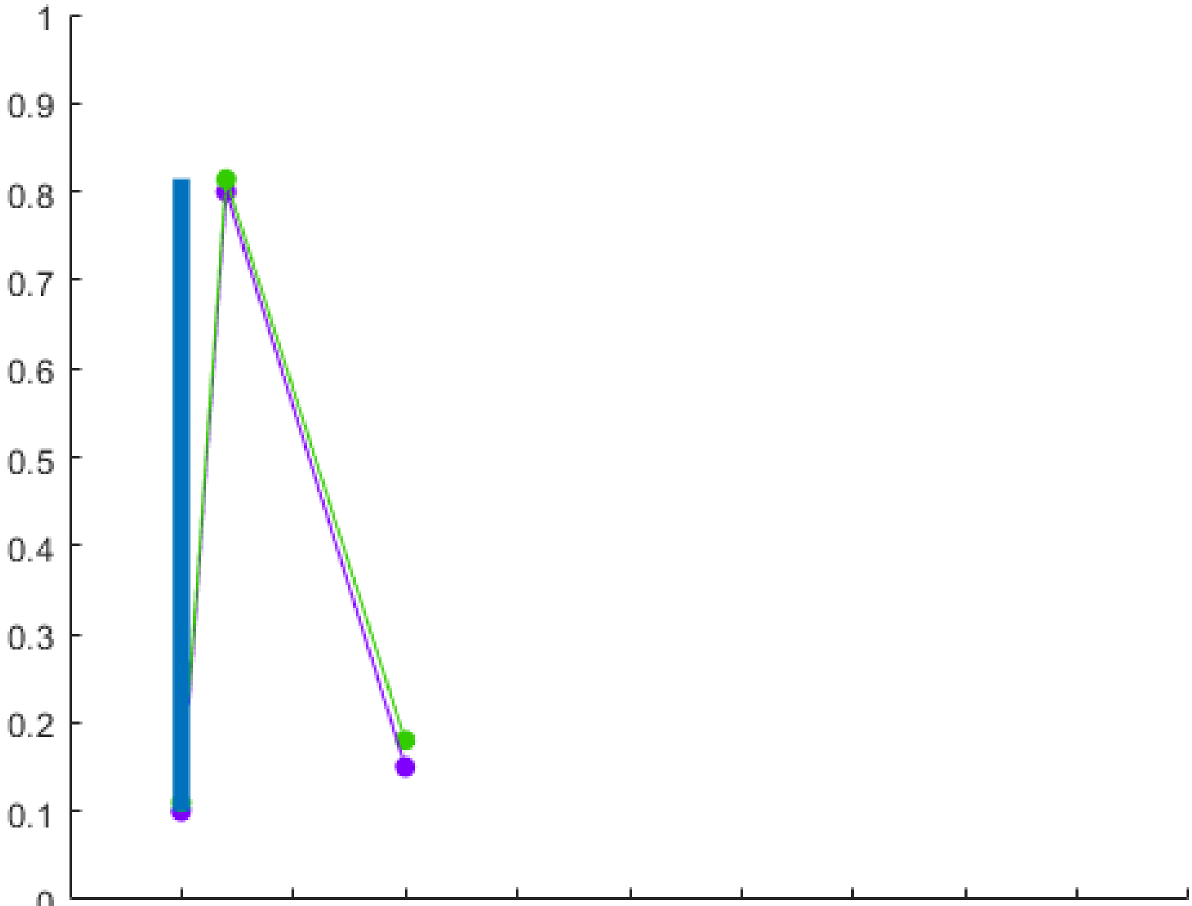


Non left-c.e. case: a construction modification

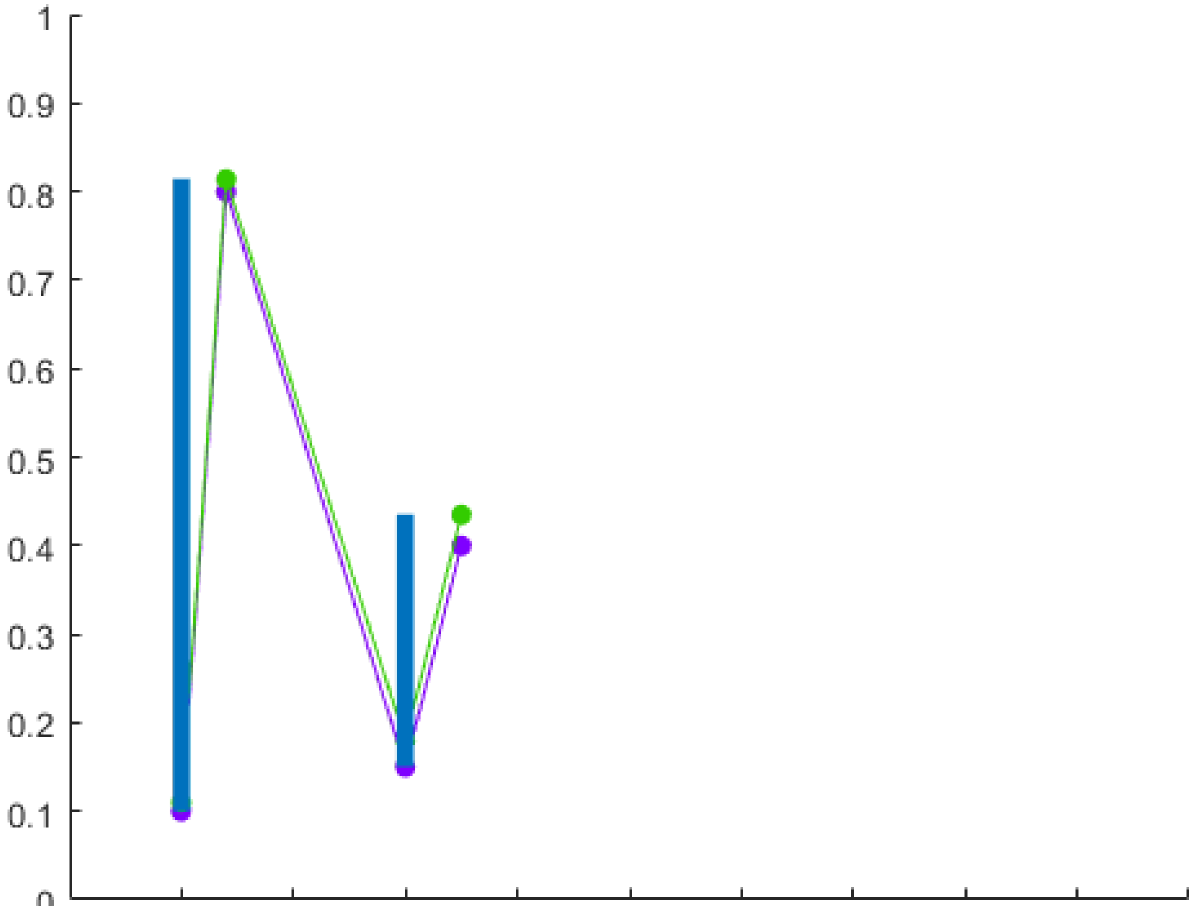
- After enumeration a new point in $dom(g)$, the new finite test may not overlap the previous one.
  $\implies$ unclear, whether we can bound uniformly the measure of the Solovay test

How to fix: at every step of original construction, after enumerating a new point q_n in $dom(g)$, we repeat iteratively the search for local minima and maxima for smaller and smaller intervals left from q_n.

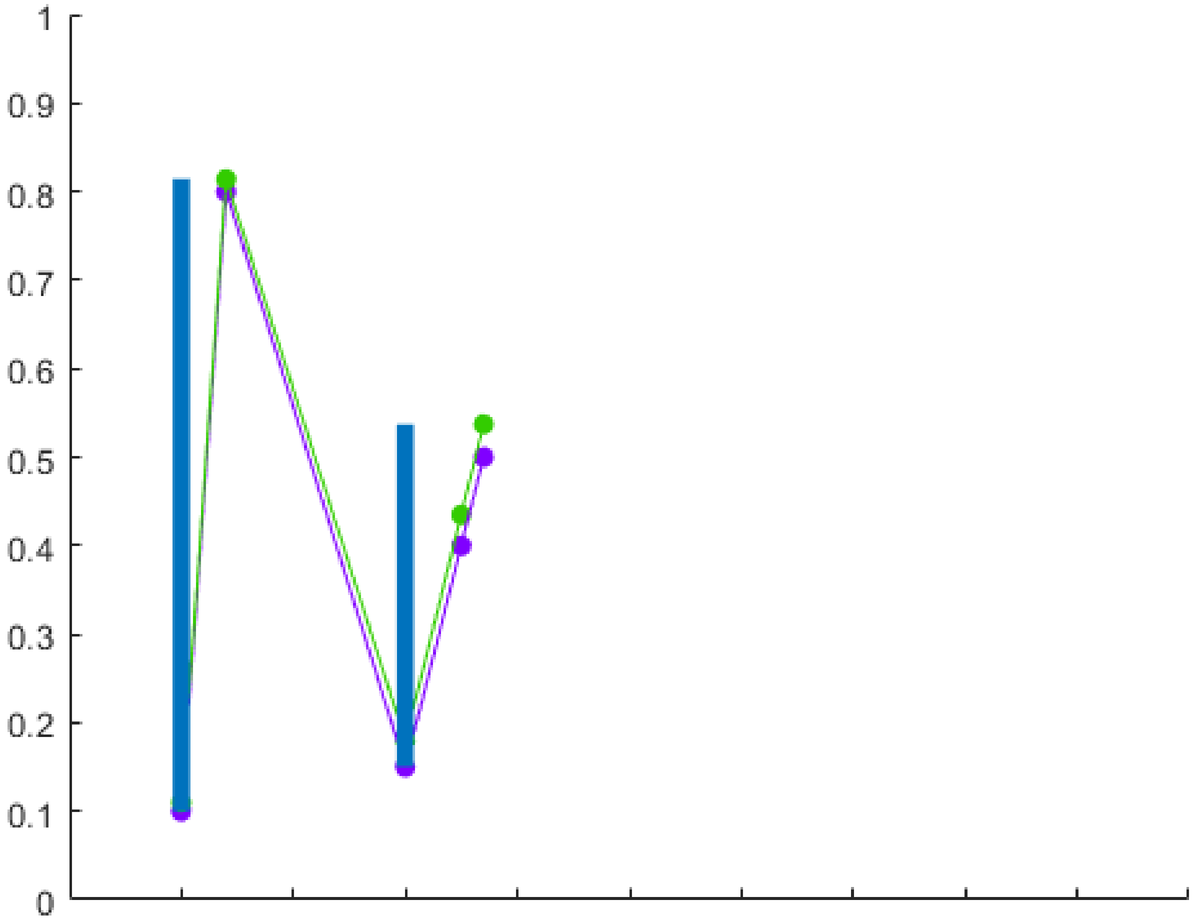
Test construction: modified



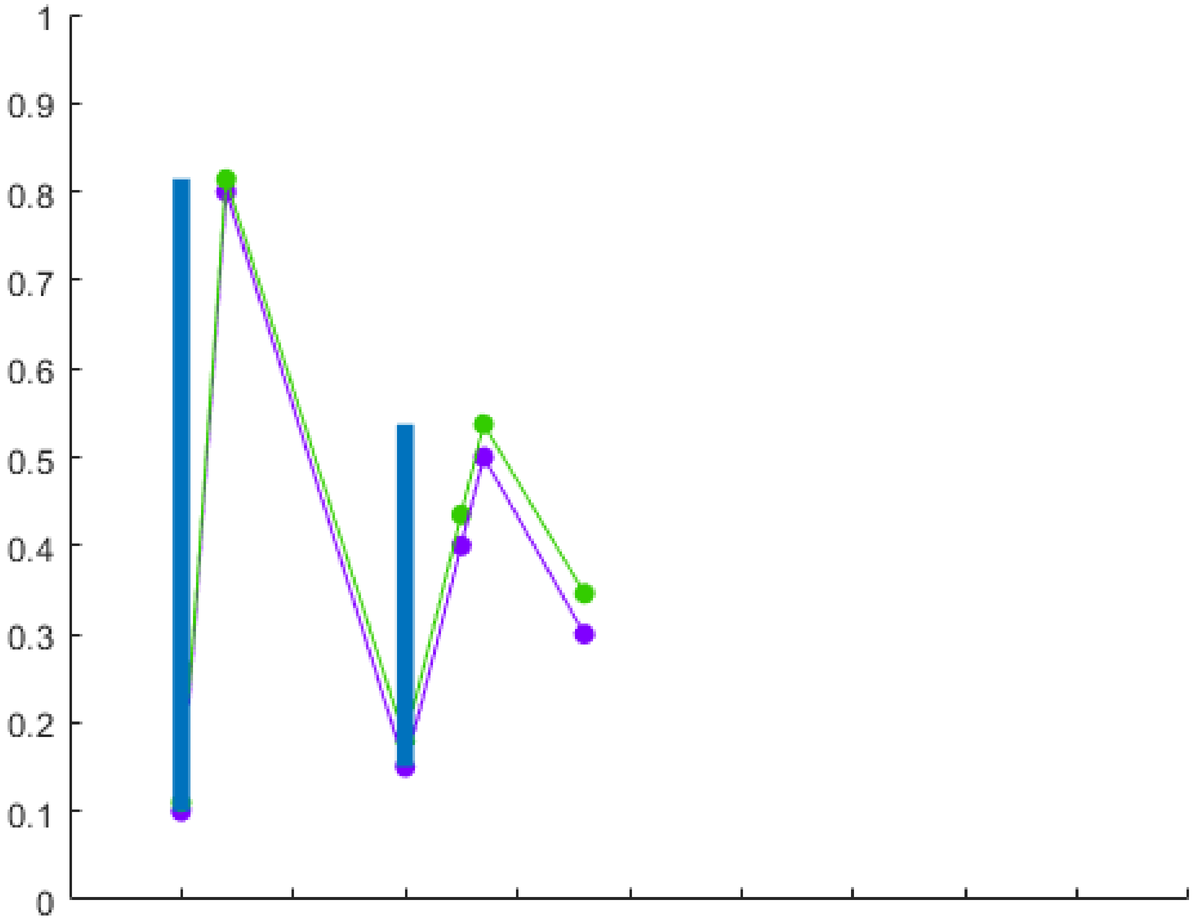
Test construction: modified



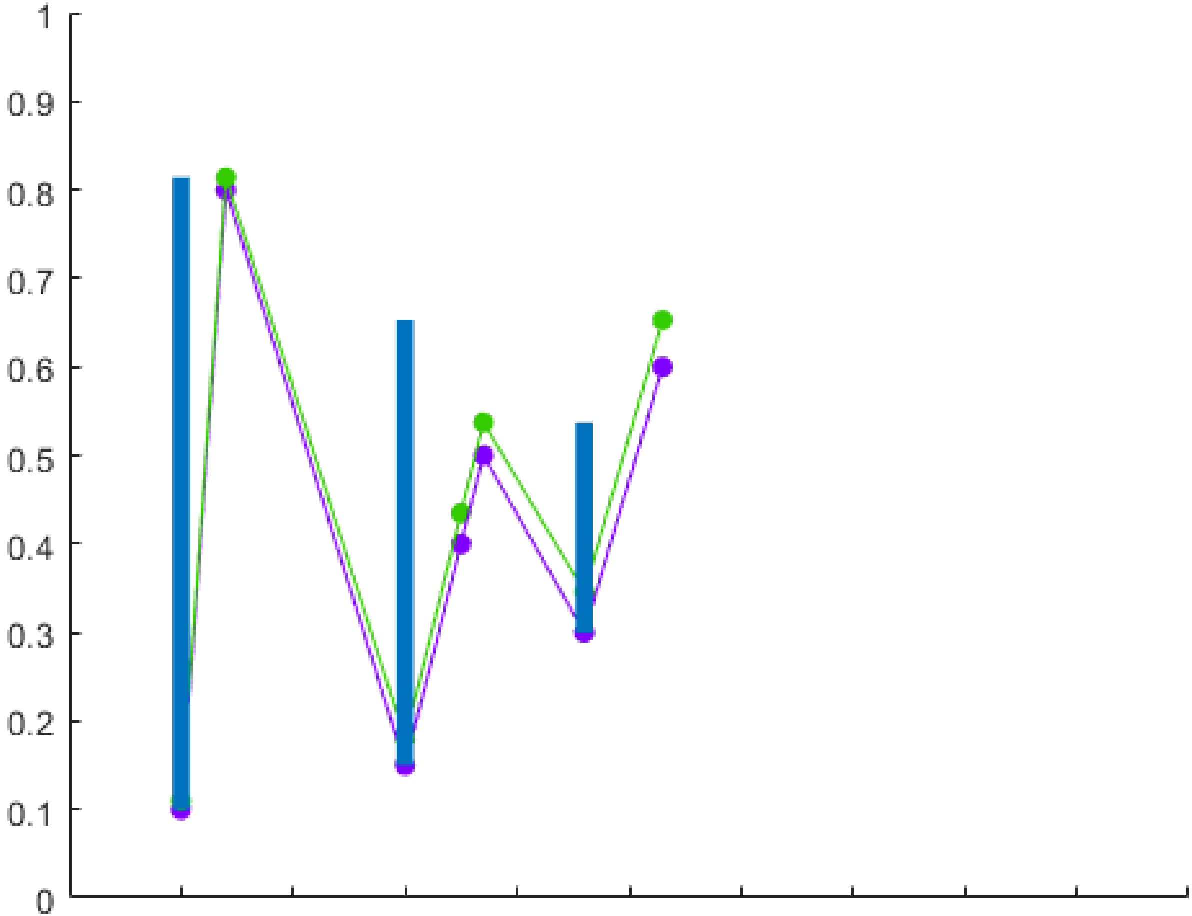
Test construction: modified



Test construction: modified



Test construction: modified



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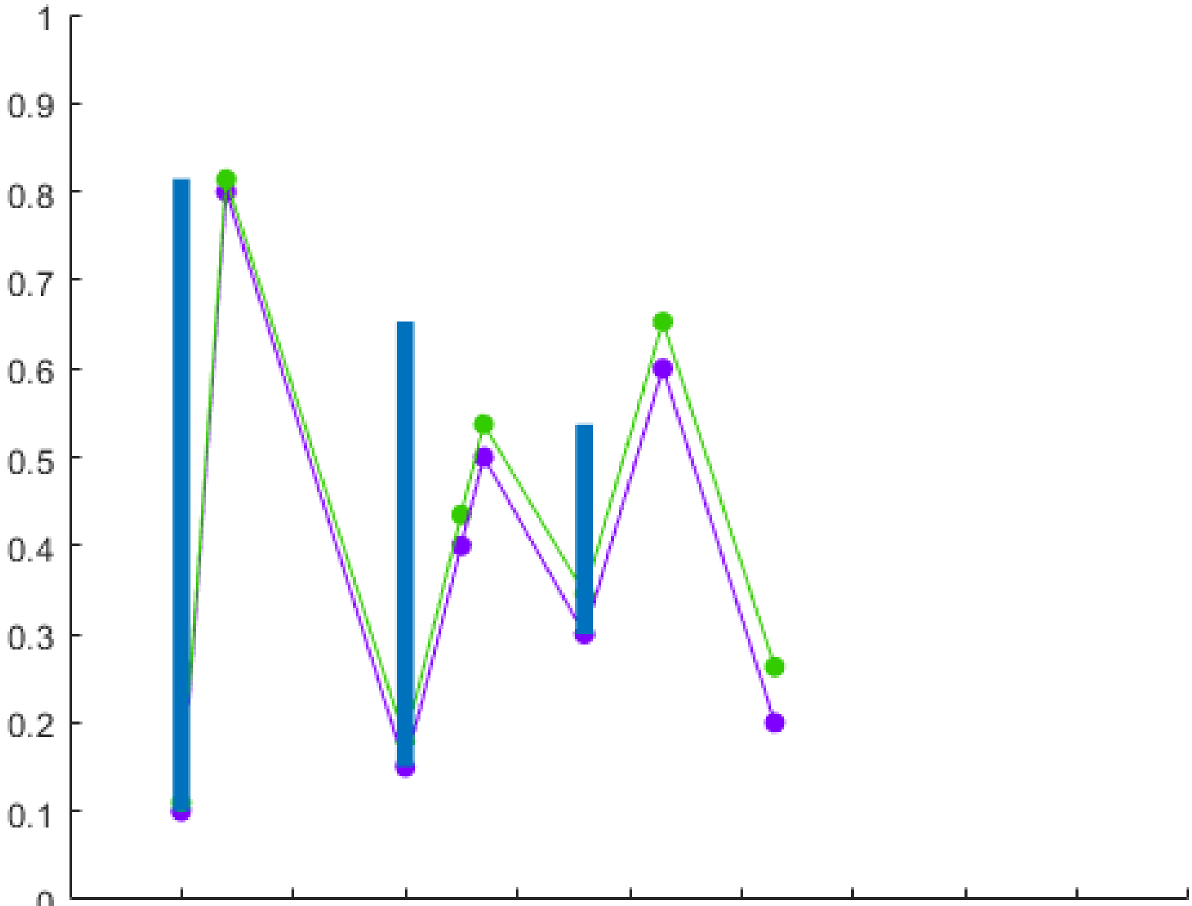

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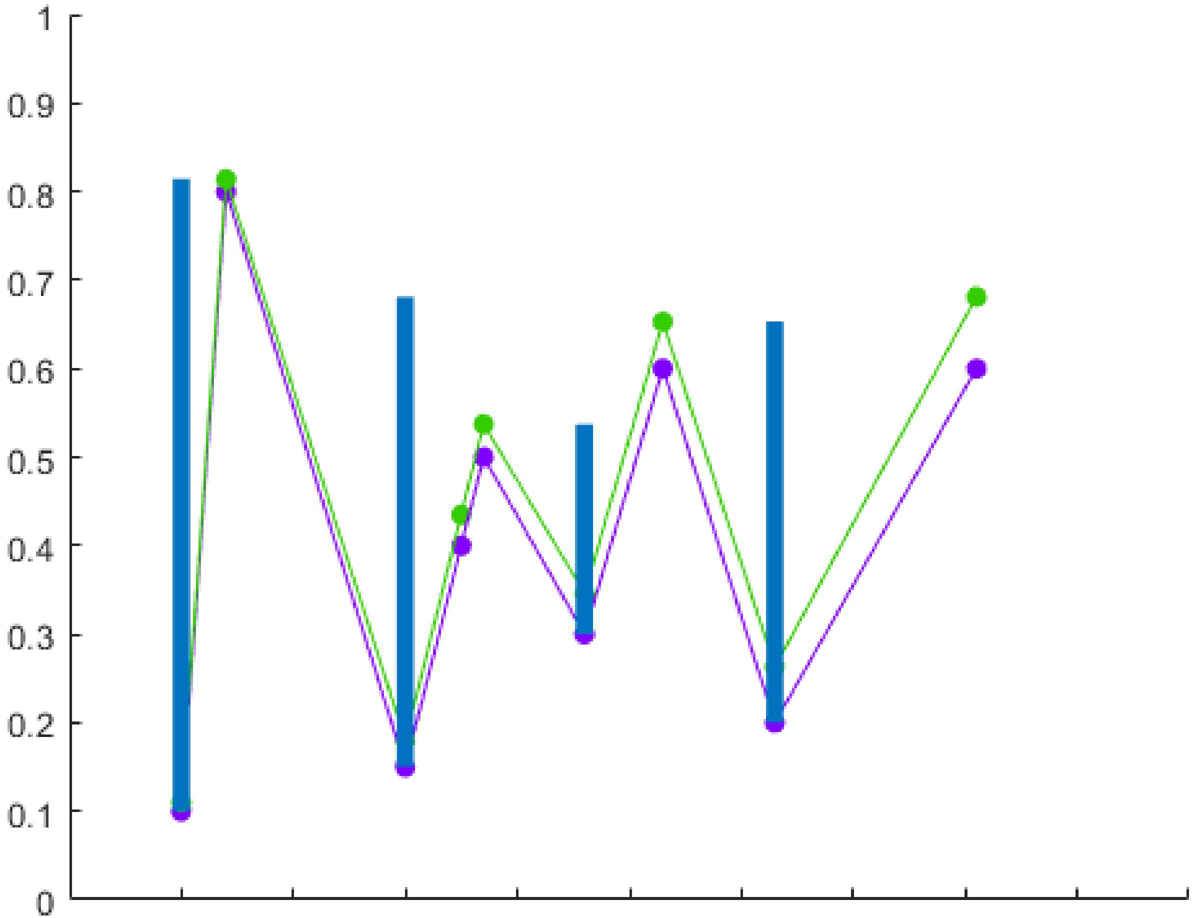

Properties of constructed test

- The test U_n constructed for the set $(q_0, \ldots, q_n)$ overlaps every test $U_{n'}$ constructed for the subset of it OR overlaps some point one times less that the subset, but in this case U_n will be overlapped in that point by every test constructed for every dataset containing $(q_0, \ldots, q_n)$. DAS KANN ICH NICHT BESSER FORMULIEREN
- Measures of U_n are still bounded for every set $(q_0, \ldots, q_n)$, hence the the measure of the Solovay test is finite.
- The Solovay test still contains $(d - c)\beta$ infinitely many times.

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