# Monotone Solovay Reducibility and Speedability

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# Outline

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# Definition

- We start by reviewing some notions of effective approximability and of randomness for real numbers as well as the "classical" notion of speedability for left-c.e. numbers.

## Definition

A **computable approximation** of a real $\alpha$ is a computable sequence $(a_n)_{n\in\omega}$ of dyadic rationals such that $\lim_{n\to\infty} a_n = \alpha$.
A **left-c.e. approximation** of a real $\alpha$ is a strictly increasing computable approximation.

A real $\alpha$ is a $\Delta^2_0$ **real** if there exists a computable approximation of $\alpha$.
A real $\alpha$ is **left-c.e.** if there is a left-c.e. approximation of $\alpha$.

# Notions of randomness

- A **Martin-Löf test** is a uniformly effective sequence of open sets $U_0, U_1, \ldots$, such that the set $U_n$ has uniform measure of at most $2^{-n-1}$.
- A **Schnorr test** is a Martin-Löf if the measures of $U_n$ are uniformly computable.
- A **Solovay test** is a uniformly effective sequence of open sets $S_0, S_1, \ldots$, such that the sum of uniform measures of all $S_n$ is finite.
- A **total Solovay test** is a Solovay test which has a computable measure.

## Definition

A sequence $A$ is **Martin-Löf random**, if there is no Martin-Löf test $U_0, U_1, \ldots$, such that $A \in \bigcap_{i \in \omega} U_i$.

A sequence $A$ is **Schnorr random**, if there is no Schnorr test $U_0, U_1, \ldots$, such that $A \in \bigcap_{i \in \omega} U_i$.

- It is well-known that a sequence $A$ is Martin-Löf random if and only if there is no Solovay test $S_0, S_1, \ldots$ such that $A$ is contained in infinitely many of the sets $S_n$ as well as Schnorr random if and only if there is no total Solovay test $S_0, S_1, \ldots$ such that $A$ is contained in infinitely many of the sets $S_n$

Further, we identify any real $\alpha := 0.A$ with its binary representation $A$.

# Speedability of left-c.e. reals

## Definition

A computable nondecreasing function $f : \mathbb{N} \to \mathbb{N}$ which fulfills $f(n) \geq n$ for every $n$ is called an **index speed-up function**.

A left-c.e. real $\alpha$ is **$\rho$-speedable with respect to its left-c.e. approximation** $(a_n)_{n\to\infty}$ for a constant $\rho \in (0,1)$ if there is an index speed-up function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\liminf_{n\to\infty} \frac{\alpha - a_{f(n)}}{\alpha - a_n} \leq \rho.$$

A left-c.e. real $\alpha$ is **$\rho$-speedable** if it is $\rho$-speedable with respect to some of its left-c.e. approximations.

# Previous results

- A left-c.e. real $\alpha$ that is $\rho$-speedable for some $\rho \in (0,1)$ is $\rho'$-speedable with respect to all of its left-c.e. approximations and for all $\rho' \in (0,1)$. (Merkle and Titov, 2020)
  This theorem allows us to use the notion of speedability on the field of left-c.e. (or right-c.e.) reals without specifying a left-c.e. (or right-c.e.) approximation and a speeding constant.
- Every Martin-Löf random left-c.e. real is nonspeedable. (Barmlalias and Lewis-Pye, 2018)
  The other direction, namely, whether there exists a nonspeedable Matrin-Löf nonrandom left-c.e. real, is still unsolved.
- Speedability is a degree property for the Solovay reducibility. (Merkle and Titov, 2020)
  Thus, if a nonspeedable Matrin-Löf nonrandom left-c.e. real exists, then all the left-c.e. reals in its Solovay degree should fulfill the same properties.

# Generalizing the notion of speedability

## Definition

A computable nondecreasing function $g :\subseteq \mathbb{Q}_2|_{[0,1)} \to \mathbb{Q}_2|_{[0,1)}$ which is defined on an interval $I$ and fulfills $f(q) \geq q$ on this interval is called a **speed-up function** on $I$.

A real $\alpha$ is $\rho$**-speedable** for a constant $\rho \in (0,1)$ if there is a speed-up function $g$ on $[0,\alpha)$ and a sequence $q_n \nearrow \alpha$ of dyadic rationals converging to $\alpha$ such that

$$\liminf_{n\to\infty} \frac{\alpha - g(q_n)}{\alpha - q_n} \leq \rho.$$

- on the set of left-c.e. reals, the new generalized definition of $\rho$-speedability is equivalent to a standard one (Merkle and Titov, 2021)

# Independence of a speeding constant

## Theorem

*A left-c.e. real $\alpha$ that is $\rho$-speedable for some $\rho \in (0,1)$ is $\rho'$-speedable for all $\rho' \in (0,1)$.*

## Proof.

(Idea) Assuming the converse, there is a real $\alpha$ and pair of rationals $\rho_1 < \rho_2$ such that $\rho_1 \approx \rho_2$ and $\alpha$ is $\rho_2$-speedable via some speed-up function but not $\rho_1$-speedable. Then, one can construct, using the information that $\frac{\alpha - g(q)}{\alpha - q} > \rho_1$ for every $q < \alpha$, a speed-up function $\tilde{g}$ which dominates $g$ and yields the $\rho_1$-speedability of $\alpha$, what contradicts the assumption. The monotony of $\tilde{g}$ follows from the inductive argument on the length of binary representation of $q$. □

# Independence of a speeding constant: proof

# Main result: nonspeedability of ML random reals

**Theorem**

*Every Martin-Löf random real is nonspeedable.*

**Proof.**

Given a speedable $\alpha$, from a (due to the previous theorem, existing) speed-up function $g$ witnessing the $\frac{1}{9}$-speedability of $\alpha$, we construct a Solovay test based on the extending of intervals $[g(q), 2g(q)]$ from left to right which overlaps $\alpha$ infinitely many times. This contradicts the Martin-Löf randomness of $\alpha$. $\square$

# Nonspeedability of ML random real: proof

# Speed-up functions and sets of limit points

**Definition**

For a real $\alpha$, a function $g : \mathbb{Q}_2|_{[0,\alpha)} \to \mathbb{Q}_2|_{[0,\alpha)}$ is called **almost accumulation point-free (aapf)** if it holds that

$$\lambda\Big(\bigcup_{q\in[0,\alpha)} [q, g(q)]\Big) = \alpha,$$

where $\lambda$ means the Lebesgue measure.

- Notice that, for a left-c.e. real $\alpha$, the speedability automatically implies the speedability via an aapf function on $[0, \alpha)$.

# The aapf property: equivalent characteristics

**Theorem**

*For every function* $g : \mathbb{Q}_2|_{[0,\alpha)} \to \mathbb{Q}_2|_{[0,\alpha)}$ *and every constant* $c \in [0, a-b]$*, the two following statements are equivalent:*

- $g$ *is aapf on* $[0, \alpha)$*,*
- $\mu\{x \in [0,\alpha) : x \notin \bigcup_{q \in [0,\alpha)} [q, g(q)]\} = \alpha$
- $\mu\{x \in [0,\alpha) : \exists q_n \nearrow x\colon\ g(q_n) \nearrow x\} = \alpha$
- *For the partial function* $g^*(x) := \lim_{q \nearrow x} g(q)$ *from* $[0, \alpha]$ *onto itself, one holds:* $\lambda\{x \in [0,\alpha) : g^*(x) \text{ is defined and } g^*(x) = x\} = \alpha.$

# BLA

**Theorem**

*Every Schnorr random real is nonspeedable via an aapf function on* $[0,1)$.

**Proof.**

In a case of a aapf speed-up function on $[0,1)$, the Solovay test constructed in the previous proof should be total since it is still disjoint while its measure is bounded by $\lambda(\bigcup_{q\in[0,1)}[q,g(q)]) = 1$ from below. □

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