Relative randomness and continuous translation functions

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Abstract

Solovay reducibility, a central object of study in algorithmic randomness, is defined via translation functions that map approximations of a real to approximations of another real. In what follows, we examine different notions of translation functions and corresponding variants of Solovay reducibility.

The main result of this article, Theorem 19, is a generalization of the Barmpalias–Lewis-Pye Limit Theorem [1] to the reducibility cl-open firstly appeared in the work of Kumabe, Miyabe, and Suzuki [5] in 2024. Furthermore, we deduce from the main result an equivalent characteristic of Martin-Löf randomness on the set of left-c.e. reals in terms of cl-open-reducibility of a real to itself.

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1 Introduction and background

Preliminaries

We start with reviewing the concept of Solovay reducibility introduced by Solovay [11] in 1975 as a measure of relative randomness and the principal results about its connection with Martin-Löf randomness on the set of left-c.e. reals. Our notation is standard. All rationals and reals are supposed to be on the interval $[0, 1)$ if not stated otherwise. A left-c.e. approximation is a strictly increasing computable approximation. Unexplained notation can be found in Downey and Hirschfeldt [2].

▶ **Definition 1** (Solovay, 1975). *A real α is* Solovay reducible *to a real β, written $\alpha \leq_S \beta$, if there exists a constant c and a partially computable function g from $\mathbb{Q}$ to $\mathbb{Q}$ such that for every $q < \beta$ it holds that*

$$0 < \alpha - g(q) \downarrow < c(\beta - q). \tag{1}$$

Solovay reducibility is meant as a measure of relative randomness in the sense that, if a real is Solovay reducible to another real, then the latter reals is considered to be at least as random as the former one. For example, already Solovay has proved [11] that the Martin-Löf random reals are closed upwards under $\leq_S$.

Solovay reducibility is nowadays considered as the "classical" standard notion of relative Martin-Löf randomness on the set of left-c.e. reals. On the latter set, Solovay reducibility has an equivalent characterization in terms of left-c.e. approximations by Calude, Hertling, Khoussainov, and Wang [10].

▶ **Proposition 2** (Calude et al., 1998). *A left-c.e. real α is Solovay reducible to a left-c.e. real β iff there exists a constant c and two strictly increasing computable sequences*

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$a_0, a_1, \dots \to \alpha$ and $b_0, b_1, \dots \to \beta$ (called LEFT-C.E. APPROXIMATIONS *of α and β, respectively, also written $a_0, a_1, \dots \nearrow \alpha$ and $b_0, b_1, \dots \nearrow \beta$) such that, for every n, it holds that*

$$0 < \alpha - a_n < c(\beta - b_n). \tag{2}$$

In 2001, Kučera and Slaman [6] demonstrated that Martin-Löf random reals form the greatest $\leq_S$-degree within the set of left-c.e. reals.

▶ **Theorem 3** (Kučera–Slaman Theorem, 2001). *Let α be a left-c.e. real and β be a Martin-Löf random left-c.e. real. Then the following statements hold. Then α is Solovay reducible to the real β.*

The latter result was strengthened in 2017 by Barmpalias and Lewis-Pye [1, Theorem 1.7] by showing a limit property of translation functions from Martin-Löf random left-c.e. reals to arbitrary left-c.e. reals.

▶ **Theorem 4** (Barmpalias–Lewis-Pye Limit Theorem 2017). *Let α be a left-c.e. real and β be a Martin-Löf random left-c.e. real. Then there exists a real constant d such that, for all left-c.e. approximations $a_0, a_1, \dots$ to α and $b_0, b_1 \dots$ to β, it holds that*

$$\lim_{n\to\infty} \frac{\alpha - a_n}{\beta - b_n} = d. \tag{3}$$

Moreover, $d = 0$ if and only if α is Martin-Löf nonrandom.

Note that Theorem 3 follows from Theorem 4.

Merkle and Titov noticed [7] that the Barmpalias-Lewis-Pye Limit Theorem applied for $\alpha = \beta$ implies that every Martin-Löf random left-c.e. real β is *nonspeedable*, i.e., that

$$\lim_{n\to\infty} \frac{\beta - b_n}{\beta - b_{f(n)}} = 1 \qquad \text{for all monotone index functions } f. \tag{4}$$

Hölzl and Janicki proved that (4) is not equivalent to Martin-Löf randomness on the set of left-c.e. reals: there exist a nonspeedable Martin-Löf nonrandom left-c.e. real that satisfies (4).

Outside of the set of left-c.e. reals, Solovay reducibility is considered by several authors as "badly behaved" [2]. Accordingly, several variants of Solovay reducibility that are better suited as relative measure of randomness in larger classes of reals have been proposed, prominently including 2aS-reducibility by Zheng and Rettinger [16].

Titov [12] proposed to use *monotone Solovay reducibility*, a variant of Solovay reducibility where the *translation function g* in Definition 1 is required to be nondecreasing. Note that Solovay reducibility agrees with its monotone variant on the set of left-c.e. reals. Titov also demonstrated that, with respect to monotone Solovay reducibility, Barmpalias–Lewis-Pye Limit Theorem can be extended to all reals.

▶ **Theorem 5** (Titov, 2024). *Let α be a real and β be a Martin-Löf random real. Then there exists a constant d such that, for every nondecreasing translation function g from β to α, it holds that*

$$\exists \lim_{q\nearrow\beta} \frac{\alpha - g(q)}{\beta - q} = d. \tag{5}$$

Furthermore, Titov showed [12] that, without the additional requirement that the translation function is nondecreasing, Theorem 5 does not hold even on the set of left-c.e. reals.

In 2020, Kumabe, Miyabe, Mizusawa, and Suzuki [4] used a similar approach, where the set of admissible translation functions is restricted to nondecreasing ones. However, they proposed to use translation functions on *reals* instead on rationals, which are required to be computable as real functions in the sense of Weihrauch [15], They introduced via this type of translation function a reducibility, which is equivalent to Solovay reducibility on the set of left-c.e. reals [4, Theorem 1]. In what follows, we refer to this reducibility as REAL SOLOVAY REDUCIBILITY and denote it by $\leq_S^{\mathbb{R}}$.

In the second and third chapters, we analyze the notions of translation functions on rationals and on reals, respectively, separately from the corresponding reducibilities, i.e., without requiring a Solovay condition like in 1.

In the fourth and fifth chapters, we prove the Barmpalias–Lewis-Pye Limit Theorem with respect to the reducibility $\leq_{cL}^{open}$ on the set of all reals.

In the sixth chapter, we discuss the notion of self-reducibility (i.e., reducibility of a real to itself) and show an equivalent characterization of Martin-Löf randomness on the set of left-c.e. reals (or, shortly, LEFT−CE) in terms of the $\leq_{cL}^{open}$-self-reducibility.

In order to distinguish the two mentioned concepts of translation functions, we will write $\mathbb{Q}$-TRANSLATION FUNCTION and $\mathbb{R}$-TRANSLATION FUNCTION for a translation function on rationals and on reals, respectively.

2 Translation functions on rationals

Titov [14] proposed to formalize the notion of a "translation function from β to α", which is a function that fulfills the conditions of Definition 1 with the distance property (1) omitted. We start the studies of Solovay reducibility by formally defining translation function on rationals and introducing the relation "there exists a translation function of rationals from β to α" between reals α and β.

▶ **Definition 6.** *A $\mathbb{Q}$-TRANSLATION FUNCTION from a real β to a real α is a partially computable function g from the set $\mathbb{Q} \cap [0, 1)$ to itself such that, for all $q < \beta$, the value $g(q)$ is defined and fulfills $g(q) < \alpha$, and it holds that*

$$\lim_{q \nearrow \beta} g(q) = \alpha, \tag{6}$$

where $\lim\limits_{q \nearrow \beta}$ denotes the left limit.

We write $\alpha \leq_{tf}^{\mathbb{Q}} \beta$ if there exists a translation function from β to α and $\alpha \leq_{tf}^{m\mathbb{Q}} \beta$ if there exists a nondecreasing translation function from β to α.

First, it is easy to see that both $\leq_{tf}^{\mathbb{Q}}$ and $\leq_{tf}^{m\mathbb{Q}}$ are reflexive and transitive; thus, they are preorders that form degree structures on $\mathbb{R}$. By [14, Corollary 1 and Proposition 3], the relation $\leq_{tf}^{\mathbb{Q}}$ is not symmetric and strictly weaker than $\leq_{tf}^{m\mathbb{Q}}$. In the next proposition, we prove the non-symmetry of $\leq_{tf}^{m\mathbb{Q}}$.

▶ **Proposition 7.** *There exist two reals α and β such that $\alpha \leq_{tf}^{m\mathbb{Q}} \beta$ but $\beta \not\leq_{tf}^{m\mathbb{Q}} \alpha$.*

Proof. See Appendix A. ◀

In other words, outside of the set of left-c.e. reals, it is not always possible to "monotonize" a $\mathbb{Q}$-translation function nor to find a monotone $\mathbb{Q}$-translation function from α to β from a given monotone $\mathbb{Q}$-translation function from β to α.

3 Translation functions on reals

In 2020, Kumabe, Miyabe, Mizusawa, and Suzuki have found [4] a characterization of Solovay reducibility on LEFT−CE that, instead of the translation functions on rationals, uses the translation functions on reals.

We start to review their approach by reminding the notion of a computable function on reals from the viewpoint of computable analysis. For unexplained notions in the next section, see [15].

3.1 Computability on the real numbers

▶ **Definition 8.** *A sequence $q_0, q_1, \dots$ of rationals is called* EFFECTIVE APPROXIMATION *if it fulfills $|q_n - q_{n+1}| < 2^{-n}$ for every n.*

Since every effective approximation $q_0, q_1, \dots$ is a Cauchy sequence, it converges to some limit point $x \in \mathbb{R}$, and, for this x, we also say that $q_0, q_1, \dots$ is a EFFECTIVE APPROXIMATION OF x.

Now, informally speaking, we call a real function f computable if there exists a machine that, for every (infinite) effective approximation of a real x in its domain, returns some (infinite) effective approximation of $f(x)$. The class of Turing machines that, using an infinite sequence of finite strings (in our case, encoded rationals) as an oracle, returns another sequence of finite strings was firstly formalized by Grzegorczyk [3] and, independently, by Lacombe [9] in 1955. For further explanations, see the monograph by Weihrauch [15, Chapter 2], where they are called "Turing machines of Type 2". In what follows, we give the formal definition of a computable real function using a notion of Turing machine of Type 2 specified for the sequences of rationals.

▶ **Definition 9** (Weihrauch, 2000). *A* TURING MACHINE M OF TYPE 2 *is an oracle Turing machine that, for every oracle $(p_0, p_1, \dots)$, where $p_0, p_1, \dots$ are (appropriately finitely encoded) rationals, produces either an infinite sequence of rationals $(q_0, q_1, \dots)$; in this case, we say that M* RETURNS THE SEQUENCE $(q_0, q_1, \dots)$ FROM THE INPUT $(p_0, p_1, \dots)$*; or a finite set of rationals $(q_0, q_1, \dots, q_n)$; in the latter case, we say that $M^{(p_0, p_1, \dots)}$* IS UNDEFINED.

A real function f from some subset of $\mathbb{R}$ to $\mathbb{R}$ is COMPUTABLE *on some set $X \subseteq \mathrm{dom}(f)$ if there exists an oracle Turing machine M such that, for every $x \in X$ and every effective approximation $p_0, p_1, \dots$ that converges to x, $M^{(p_0, p_1, \dots)}$ returns an effective approximation $(q_0, q_1, \dots)$ of $f(x)$.*

By [15, Corollary 4.3.1], computability on reals implies continuity.

▶ **Proposition 10.** *Every real function, which is computable on some interval $[a, b]$, is continuous in every point in (a, b).*

The following proposition is straightforwardly implied by [15, Corollary 6.2.5].

▶ **Proposition 11.** *If a real function g is computable on the set $[a, b)$, then the maximum function $h(x) = \max\{g(y) : a \leq y \leq x\}$ is computable.*

3.2 Variants of translation functions on reals

Kumabe, Miyabe, Mizusawa, and Suzuki characterized [4, Theorem 1] the Solovay reducibility on LEFT−CE using different versions of translation functions on reals.

In the similar way as for translation functions on rationals, we start to explore these reducibilities by introducing the notions of an "$\mathbb{R}$-translation funciton" and a "weakly $\mathbb{R}$-translation function" formalized by Titov [14], which do not require neither monotonicity nor any distance conditions.

▶ **Definition 12.** *A* WEAKLY $\mathbb{R}$-TRANSLATION FUNCTION *from a real β to a real α is a real function f which is computable on the interval $[0, \beta)$ and satisfies*

$$\lim_{x \nearrow \beta} f(x) = \alpha. \tag{7}$$

An $\mathbb{R}$-TRANSLATION FUNCTION *from a real β to a real α is a weakly $\mathbb{R}$-translation function from β to α that maps $[0, \beta)$ to the interval $[0, \alpha)$. We write $\alpha \leq_{\text{tf}}^{\mathbb{R}} \beta$ if there exists a $\mathbb{R}$-translation function from β to α.*

It is easy to see that $\leq_{\text{tf}}^{\mathbb{R}}$ is a preorder that induces a degree structure. In contrast to $\leq_{\text{tf}}^{\mathbb{Q}}$, the restriction of admissible $\mathbb{R}$-translation functions to only nondecreasing ones does not induce any stricter relation, as we will see in the next proposition.

▶ **Proposition 13.** *Let f be a computable $\mathbb{R}$-translation function on reals from β to α. Then, the function $h(x) = \max\{f(y) : y \leq x\}$ is a nondecreasing computable $\mathbb{R}$-translation function from β to α.*

Proof. The function h defined as in the proposition statement is computable (in the sense of Definition 9) by Proposition 11. Moreover, h is obviously nondecreasing, and it holds that

$$\text{for every } x < \beta, \text{ there exists } y \in [0, \beta) \text{ such that } h(x) = f(y) < \alpha. \tag{8}$$

Moreover, we have $\lim_{x \nearrow \beta} h(x) = \alpha$ by the following argument:

- $\lim_{x \nearrow \beta} h(x) = \alpha$ since $\liminf_{x \nearrow \beta} h(x) \geq \liminf_{x \nearrow \beta} f(x) = \lim_{x \nearrow \beta} f(x) = \alpha$ by (7);
- $\limsup_{x \nearrow \beta} h(x) \leq \alpha$ by (8).

Thus, h is a nondecreasing $\mathbb{R}$-translation function from β to α. ◀

3.3 Reducibilities defined via translation function on reals

Kumabe et al. introduced in 2020 [4, Definition 9] and 2024 [5, Definition 5.1] two new types of reducibility by requiring the $\mathbb{R}$-translation functions and weakly $\mathbb{R}$-translation functions, respectively, to be Lipschitz continuous (note that replacing the requirement (1) in the definition of Solovay reducibility by the requirement for the $\mathbb{Q}$-translation g to be Lipschitz continuous yields a reducibility notion which is equivalent to $\leq_{\text{S}}$ on LEFT−CE; for a formal proof of this fact, see [14, Proposition 1]). In what follows, we give the formal definition of these reducibility; where the reducibility denoted by Kumabe et al. as "L2" obtains a more intuitive name "real Solovay reducibility".

▶ **Definition 14** (Kumabe et al., 2020; Kumabe et al., 2024)**.** *A real α is* REAL SOLOVAY REDUCIBLE *to a real β, written $\alpha \leq_{\text{S}}^{\mathbb{R}} \beta$, if there exists a Lipschitz continuous $\mathbb{R}$-translation function from β to α.*

A real α is CL-OPEN REDUCIBLE *to a real β, written $\alpha \leq_{\text{cL}}^{\text{open}} \beta$, if there exists a Lipschitz continuous weakly $\mathbb{R}$-translation function from β to α.*

Titov [14, Corollary 3] has also shown that the additional requirement for the $\mathbb{R}$-translation function in the latter definition to be nondecreasing (Kumabe et al. denoted [4, Definition 9] the resulting reducibility "L1") does not induce any strictly stronger reducibility on $\mathbb{R}$ than $\leq_{\text{S}}^{\mathbb{R}}$.

▶ **Proposition 15** (Titov, 2025). *If* $\alpha \leq_{\mathrm{S}}^{\mathbb{R}} \beta$ *for two reals* α *and* β*, then* $\alpha \leq_{\mathrm{S}}^{\mathbb{R}} \beta$ *via a nondecreasing* $\mathbb{R}$*-translation function.*

In the latter definitions, the Lipschitz continuity requirement can be replaced by a "localized" version of Lipschitz continuity, where instead of arbitrary pairs of arguments, we consider only pairs with second component β, similarly to the inequality (1) for translation functions on rationals.

▶ **Proposition 16.** *For all reals* α *and* β*, the following equivalences hold:*

1. $\alpha \leq_{\mathrm{S}}^{\mathbb{R}} \beta$ *iff there exists a constant* c *an* $\mathbb{R}$*-translation function* f *from* β *to* α *that fulfills*

$$\frac{\alpha - f(x)}{\beta - x} < c. \tag{9}$$

2. $\alpha \leq_{\mathrm{cL}}^{\mathrm{open}} \beta$ *iff there exists a constant* c *and a weakly* $\mathbb{R}$*-translation function* f *from* β *to* α *that fulfills*

$$\frac{|\alpha - f(x)|}{\beta - x} < c. \tag{10}$$

Proof. Direction $\Longrightarrow$ is straightforward in both equivalences since every Lipschitz continuous weakly $\mathbb{R}$-translation function f from β to α fulfills (10) with the value of c greater than Lipschitz constant of f. In case f maps $[0, \beta)$ to $[0, \alpha)$, we also obtain (9) from (10) since it holds $|\alpha - f(x)| = \alpha - f(x)$ for all $x \in [0, \beta)$.

The inverse directions will be obtained for every equivalence separately: first, let f be an $\mathbb{R}$-translation function f that fulfills (9). Then the function $g(x) = \min\{f(y) : y \leq x\}$ is, by Proposition 13, is a nondecreasing $\mathbb{R}$-translation function from β to α that still fulfills (9). Hence, the function $h(y) = \min\{g(y) + c(y - x) : y \leq x\}$ witnesses $\alpha \leq_{\mathrm{S}}^{\mathbb{R}} \beta$.

For a weakly $\mathbb{R}$-translation function $\tilde{f}$ that fulfills (9), we define

$$\begin{aligned} \tilde{g}(x) &= \min\{\tilde{f}(y) + d(x - y) : y \in [0, x]\} \quad \text{and} \\ \tilde{h}(x) &= \max\{\tilde{g}(y) - d(x - y) : y \in [0, x]\}. \end{aligned}$$

Then $\tilde{h}$ is a weakly $\mathbb{R}$-translation function that witnesses $\alpha \leq_{\mathrm{cL}}^{\mathrm{open}} \beta$. ◀

We will refer to the equalities (9) and (10) as Solovay inequality and weak Solovay inequality, respectively.

By [4, Theorem 1] and by [5, Observation 5.3], respectively, the reducibilities $\leq_{\mathrm{S}}^{\mathbb{R}}$ and $\leq_{\mathrm{cL}}^{\mathrm{open}}$ are equivalent to the Solovay reducibility $\leq_{\mathrm{S}}$ on the set of left-c.e. reals, and, by [14, Theorem 1], there are implied by $\leq_{\mathrm{S}}$ on $\mathbb{R}$ and $\mathbb{R} \setminus \mathrm{COMP}$, respectively.

▶ **Theorem 17** (Titov, 2025). *The reducibility* $\leq_{\mathrm{S}}$ *implies* $\leq_{\mathrm{S}}^{\mathbb{R}}$ *on all but computable reals and* $\leq_{\mathrm{cL}}^{\mathrm{open}}$ *on all reals.*

4 Prelimilaries to Kučera-Slaman Theorem and Barmpalias-Lewis-Pye Limit Theorem for weakly $\mathbb{R}$ translation functions

4.1 Previous results

The characterization of Solovay reducibility given in Proposition 2 gave a new start to the research the Solovay reducibility because of its close connection with the construction of Martin-Löf and Solovay randomness tests.

In 2017, Miller divided [8] the Barmpalias-Lewis-Pye Limit Theorem into two logically independent statements that we denote by (**KS**) and (**BLP**) in what follows.

(KS) If there are two left-c.e. approximations $a_0, a_1, \dots$ and $b_0, b_1, \dots$ of a left-c.e. real α and a Martin-Löf random left-c.e. real β, respectively, then they witness the Solovay reducibility $\alpha \leq_{\mathrm{S}} \beta$ in the sense of Proposition 2.

This can be considered as a reformulation of Kučera–Slaman Theorem (which motivates the choice of the acronym "KS"), even quite strengthened since the Solovay reducibility is witnessed by *every* pair of left-c.e. approximations of α and β, respectively. For a direct proof of this statement, see [8, Lemma 1.1].

(BLP) For every left-c.e. real α and Martin-Löf random left-c.e. real β, there exists a constant d such that, for every left-c.e. approximations $a_0, a_1, \dots \nearrow \alpha$ and $b_0, b_1, \dots \nearrow \beta$ that witness the Solovay reducibility $\alpha \leq_{\mathrm{S}} \beta$ in the sense of Proposition 2 with some constant c, (3) holds.

Moreover, $d = 0$ if and only if α is Martin-Löf nonrandom.

For the direct proof of this statement, see [8, Lemma 1.2]. It is easy to see that both statements together imply exactly Theorem 4.

In the remainder of this chapter, we state each of them on $\mathbb{R}$ relative to $\leq_{\mathrm{cL}}^{\mathrm{open}}$ using the concept of total variation of a function on reals that we recall in what follows.

4.2 Total variation

▶ **Definition 18.** *A* FINITE PARTITION *of an open interval (a,b) is a tuple of finitely many reals reals $(x_0, \dots, x_n)$ where $n \in \mathbb{N}$ such that*

$$a < x_0 < \dots < x_n < b. \tag{11}$$

A FINITE PARTITION *of the interval $[a,b)$, $(a,b]$, or $[a,b]$ is a tuple $(x_0, \dots, x_n)$ where $n \in \mathbb{N}$ that fulfills (11) with first $<$, last $<$, or both first and last $<$, respectively, replaced by $\leq$.*

The TOTAL VARIATION *$V_I(f)$ of a real function $f : I \to \mathbb{R}$, where I is an open, semi-open or closed interval, is the quantity*

$$V_I(f) = \sup_{(x_0,\dots,x_n)\in\mathcal{P}} \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)|, \tag{12}$$

where $\mathcal{P}$ is the set of all partitions of I.

We say that f has a FINITE TOTAL VARIATION *if $M < \infty$, and* INFINITE TOTAL VARIATION *otherwise.*

The next analytical lemma shows the uniformity of the notion of total variation in the sense that the supremum in (12), can be approximated from below using an arbitrary enumeration of every dense subset of I.

▶ **Lemma 19.** *Let I be an interval, $f : I \to \mathbb{R}$ be a continuous real function with a finite total variation $V_f < \infty$. Then, for every enumeration $x_0, x_1, \dots$ of an arbitrary countable dense subset $X \subseteq I$, it holds that*

$$V_f = \lim_{n\to\infty} \sum_{i=0}^{n-1} \left| f(y_{i+1}^n) - f(y_i^n) \right|, \tag{13}$$

where $y_0^n < \dots < y_n^n$ is the set $\{x_0, \dots, x_n\}$ sorted increasingly.

Proof. See Appendix B. ◀

5 Barmpalias-Lewis-Pye Theorem for weakly $\mathbb{R}$-translation functions

▶ **Theorem 20.** *Let α and β be reals such that β is Martin-Löf random. Then for every weakly $\mathbb{R}$-translation function f from β to α with a finite total variation, it holds that*

$$\lim_{x \nearrow \beta} \frac{\alpha - f(x)}{\beta - x} = d. \tag{14}$$

Moreover, in case such function f exists and $d \neq 0$, then α is Martin-Löf random as well, and it holds $\alpha \equiv^{\mathbb{R}}_{\mathrm{S}} \beta$.

▶ Remark 21. In Theorem 20, the requirement for the function f cannot be omitted even for left-c.e. reals α and β. Indeed, for every left-c.e. approximation $b_0, b_1, \ldots$ of a Martin-Löf random left-c.e real β, one can easily construct an $\mathbb{R}$-translation function from β to β, such that

$$\limsup_{x \nearrow \beta} \frac{\beta - f(x)}{\beta - x} \geq \limsup_{n \to \infty} \frac{\beta - f(b_n)}{\beta - b_n} = \infty,$$

hence the theorem statement (14) for the function f and $\alpha = \beta$ does not hold. We leave the exact construction as an exercise for the reader.

Proof. Let α and β be two real on $[0, 1)$, and let f be a weakly $\mathbb{R}$-translation function from β to α computed by a Turing machine M of Type 2 with a finite total variation $V_I(f)$ on the biggest interval I on that f is totally computable (in particular, $[0, \beta) \subseteq I$).

We organize the proof in the manner of Miller [8] by showing consequently that the ratio in (14) is bounded, that this ratio has a left limit in β, and that this limit does not depend on the choice of f. Finally, we consider the case $d = 0$.

Outline of the proof

1. In Paragraph 5.1, we define two functions f^- and f^+ from rationals to rationals that have the same domain and return the value of f with a convenient precision.
2. In Paragraph 5.2, we prove that f fulfills weak Solovay inequality (10) as follows: using f^- and f^+, we define two Martin-Löf tests S and T whose every level will be constructed by computing in every next enumeration step of the domain of f^- and f^+ a finite test that extends a test constructed in the previous step. Then β should pass both tests by its Martin-Löf randomness, hence, for some i, the i^{th} levels of both tests do not contain β. Due to the specific construction of the tests, it will imply the weak Solovay inequality for f with an appropriate Solovay constant. Finally, we turn f into a Lipschitz continuous weakly $\mathbb{R}$-translation function from β to α. During the whole construction, we will use technical claims whose proofs will be given separately in Appendix C.
   Note that, by Proposition 16, this part implies as a corollary that $\alpha \leq^{\text{open}}_{\text{cL}} \beta$.

   The part proves a generalization of the statement (**KS**) on all reals for weakly $\mathbb{R}$-translation functions. On left-c.e. reals, it easily implies (**KS**).
3. In Paragraph 5.3, we prove that, for every function f that fulfills (10), there exists a constant d that fulfills (14) by contradiction: supposing the converse, we define two functions f^- and f^+ from rationals to rationals that have the same domain and return values of f with convenient precision. Then we define a Solovay test S whose every level will be a finite test computed in every next enumeration step of the domain of f^- and f^+ that extends a test computed in the previous step. Due to the specific construction of

the test, it will imply that either contain β infinitely many times, which contradicts its Martin-Löf randomness.

This is the main part of the proof, which is based on the proof of [13, Theorem 66] adapted to real-valued translation functions. Due to its volume as well as its strong similarity to the proof of mentioned Theorem 66, this proof has been completely moved to Appendix D.

4. In Paragraph 5.4, we prove the uniqueness of d by contradiction: supposing the existence of two different functions f_1 and f_2 witnessing $\alpha \leq^{\text{open}}_{\text{cL}} \beta$ with the different values of d, we obtain an effective approximation of β (which cannot exist for Martin-Löf random reals) by using the computable function $f_1 - f_2$. This part is directly implied by properties of computable real-valued functions.

5. Finally, in Paragraph 5.5 in case $d \neq 0$, we construct a function $\tilde{f}$ witnessing $\beta \leq_{\text{S}} \alpha$ by inverting f. This implies the Martin-Löf randomness of α since, by [14, Proposition 9(a)], Martin-Löf random reals are $\leq^{\text{open}}_{\text{cL}}$-closed upwards in $\mathbb{R}$. This part has no analogues for Solovay reducibility via rational translation functions because, by Proposition 7, strictly increasing translation functions on rationals are not always invertible.

Note that the latter case also implies $\alpha \equiv^{\mathbb{R}}_{\text{S}} \beta$.

The latter three parts together build a generalization of the statement (**BLP**) on all reals for weakly $\mathbb{R}$-translation functions. On left-c.e. reals, they easily imply (**BLP**).

Notation

In the remainder of this proof and unless explicitly stated otherwise, for two reals $a < b$, the length of the interval $I = [a, b]$ will be denoted by $|I|$, and the notation $[b, a]$ will denote an empty interval with length 0. A TEST SET is tuple of pairs of rationals $Q = \left(\begin{pmatrix} r_0 \\ s_0 \end{pmatrix}, \ldots, \begin{pmatrix} r_n \\ s_n \end{pmatrix}\right)$ if it fulfills the inequality $s_0 \leq s_1 \cdots \leq s_n$, and a FINITE TEST an empty set or a tuple $A = (U_0, \ldots, U_m)$ with $m \geq 0$ where the U_i are not necessarily distinct nonempty intervals.

We also say that a finite test A IS COVERED by another finite test B, written $A \preceq B$, if the union of all intervals contained in the tuple A is covered by the union of all intervals contained in the tuple B:

$$A \preceq B \quad :\Longleftrightarrow \quad \bigcup_{I \in A} I \subseteq \bigcup_{I \in B} I. \tag{15}$$

5.1 From functions on reals to functions on rationals

Let $p_0, p_1, \ldots$ be an enumeration of all rationals in $[0, 1)$, and define a two-argument-function $g :\subseteq \mathbb{Q}|_{[0,1)} \times \mathbb{N} \to \mathbb{Q}$ as follows: for every p, let

$$g(p, n) = r_{2n} \text{ if } \begin{cases} (M^{(p,p,\ldots)} \upharpoonright 2n) \downarrow = (r_1, \ldots, r_{2n}) \\ r_{i+1} - r_i < 2^i \text{ for every } i \in \{1, \ldots, 2n-1\} \end{cases}$$

(remind that M is a Turing machine of Type 2 that computes f) and $(p, n) \uparrow$ otherwise. In particular, if the value of $f(p)$ is defined (which holds, inter alia, for all rationals in I), then, for all n, $g(p, n)$ halts and returns it with accuracy 2^{-2n}.

Next, let $q_0, q_1, \ldots$ be the a sub-sequence of $p_0, p_1, \ldots$ obtained by the following dovetailing: At the step i, add into the sequence $(q_0, q_1, \ldots)$ some p_k such that $g(p_k, 2i)$ is defined and the sequence $(q_0, q_1, \dots)$ does still not contain p_k. By the previous discussion, all rationals in I will be enumerated into $q_0, q_1, \dots$, hence the functions

$$\tilde{f}(q) = g(q_n, 2n) \quad \text{and} \quad \begin{cases} f^-(q) = \tilde{f}(q_n) - 2^{-2n} \\ f^+(q) = \tilde{f}(q_n) + 2^{-2n} \end{cases} \quad \text{if there exists } n \text{ such that } q = q_n \tag{16}$$

are defined for all q_n, in particular, on the whole set $\mathbb{Q}|_I$.

It obviously holds for every n that

$$|\tilde{f}(q_n) - f(q_n)| = |g(q_n, 2n) - f(q_n)| \leq 2^{-2n}, \text{ hence } f^-(q_n) \leq \tilde{f}(q_n) \leq f^+(q_n), \tag{17}$$

wherein the sum of distances between $f^+(q)$ and $f^-(q)$ for all their arguments is bounded from above:

$$\sum_{n \in \mathbb{N}} |f^+(q_n) - f^-(q_n)| \leq \sum_{n \in \mathbb{N}} 2 \cdot 2^{-2n} = \frac{8}{3}. \tag{18}$$

The next two claims describe the further properties of functions $\tilde{f}$, f^+, and f^-. The first one follows from Lemma 19 and (17), the second one from (17) and density of both functions on $[0, \beta)$.

▷ Claim 22. Let n be a natural, and let the tuple $(q_0^n, \dots q_n^n)$ be the set $\{q_0, \dots, q_n\}$ sorted increasingly. Then we have

$$\sum_{i=0}^{n-1} |\tilde{f}(q_{i+1}^n) - \tilde{f}(q_i^n)| \leq M + \frac{8}{3} < \infty. \tag{19}$$

▷ Claim 23. The value α is the left limit of function $\tilde{f}$ in β, i.e.,

$$\exists \lim_{q \nearrow \beta} \tilde{f}(q) = \alpha. \tag{20}$$

5.2 f fulfills the weak Solovay inequality

Now, using the functions $\tilde{f}$ and g, we construct two Martin-Löf tests S and T by applying in every enumeration step of $q_0, q_1, \dots$ the same algorithm that will be described in what follows on an appropriate test set.

A single stage of the construction

Fix a natural i (called LEVEL) and a test set $Q = (\binom{a_0}{b_0}, \binom{a_1}{b_1}, \dots, \binom{a_n}{b_n})$ where $n \geq 0$.

We describe the construction of the finite test $Test_i(Q)$, which is a lightly modified version of a construction used by Titov [12, Theorem 2.1].

For every two indices k, m, such that $0 \leq k < m \leq n$, define the interval

$$I[k, m] := R[\binom{a_k}{b_k}, \binom{a_m}{b_m}] := \begin{cases} [b_m, b_k + \frac{a_m - a_k}{2^{i+1}}] & \text{if } \frac{a_m - a_k}{b_m - b_k} \geq 2^{i+1}, \\ \emptyset & \text{otherwise,} \end{cases} \tag{21}$$

and put the intersection of $I[k, m]$ with the unit interval $[0, 1]$ into the test $Test_i(Q)$, obtaining

$$Cover(Test_i(Q)) = \left(\bigcup_{k, m \in \{0, \dots, n\} : k < m} I[k, m] \right) \cap [0, 1]. \tag{22}$$

Further, for technical reasons, for all k, m such that $0 \leq m \leq k \leq n$, set $I[k, m] = \emptyset$. We will also write $I^i[k, m]$ for $I[k, m]$ in case the level i is not clear from the context.

Then, this construction fulfills the inclusion properties by levers and test sets, respectively, described in the next claim.

▷ Claim 24. For every test set P and every two indices i and j, $i < j$ implies that $Test_j(P) \preceq Test_i(P)$.

For every index i and every two test sets P and Q, $P \subseteq Q$ implies that $Test_i(P) \preceq Test_i(Q)$ for every index i.

The claim describes the inclusion properties of particular intervals of $Test_i(Q)$.

▷ Claim 25. For every two indices k and l where $0 \leq k < l \leq n$, the following implications hold:

$$\frac{a_l - a_k}{b_l - b_k} \leq 2^{i+1} \quad \Longrightarrow \quad \forall m > l\big(I[k, m] \subseteq I[l, m]\big), \tag{23}$$

$$\frac{a_l - a_k}{b_l - b_k} \geq 2^{i+1} \quad \Longrightarrow \quad \forall m\big(I[k, m] \supseteq I[l, m]\big). \tag{24}$$

In what follows, we defined an index subset $i_0, \ldots, i_s$ of $0, \ldots, n$ and see that every level $Test_i(Q)$ of $Test(Q)$ is a disjoint union of some intervals defined via $i_0, \ldots, i_s$.

Let $0 = i_0 < i_1 < \cdots < i_s$ be the indices in the range $0, \ldots, n$ such that

$$\frac{a_m - a_{i_s}}{b_m - b_{i_s}} > 2^{i+1} \quad \text{for all} \quad m \in \{i_s, \ldots, n\} \tag{25}$$

and, for every $j \in \{0, \ldots, s-1\}$,

$$\frac{a_{i_{j+1}} - a_{i_j}}{b_{i_{j+1}} - b_{i_j}} \quad \leq \quad 2^{i+1}, \tag{26}$$

$$\frac{a_m - a_{i_j}}{b_m - b_{i_j}} \quad > \quad 2^{i+1} \quad \text{for all} \quad m \in \{i_j + 1, \ldots, i_{j+1} - 1\}. \tag{27}$$

Further, due to the technical reasons, we fix an additional index $i_{s+1} = n + 1$ (hence $i_{s+1} - 1 = n$) and set $q_{n+1} = 1$ and $g(q_{n+1}) = 1$, so (25) is nothing but (27) for $j = s$.

Next, define for every j from 0 to s the index h_j in the range $i_j, \ldots, s$ such that

$$a_{h_j} = \max\{a_h : i_j \leq h < i_{j+1}\}. \tag{28}$$

▷ Claim 26. If a real x is not contained in any interval $[b_{i_j}, b_{i_j} + \frac{a_{h_j} - a_{i_j}}{2^{i+1}}]$ for all j in the range $0, \ldots, s$, then we have $x \notin I[k, m]$ for all k and m.

By (22) the latter claim implies as a corollary that

$$Cover(Test(Q_i)) \subseteq \bigcup_{j \in \{0, \ldots, s\}} [b_{i_j}, b_{i_j} + \frac{a_{h_i} - a_{i_j}}{2^{i+1}}]. \tag{29}$$

Construction and properties of two Martin-Löf tests S and T

We will construct two Martin-Löf tests T with levels $S_0, S_1, \ldots$ and T with levels $T_0, T_1, \ldots$ iteratively: first, at stage 0, we initialize for every natural i the levels $S_0^0, S_i^0, \ldots$ and $T_0^0, T_1^0, \ldots$ as empty sets; in every next stage $n > 0$, we define for every natural i the test sets

$$Q_n = (\begin{pmatrix} f^-(q_0^n) \\ q_0^n \end{pmatrix}, \begin{pmatrix} f^+(q_0^n) \\ q_0^n \end{pmatrix}, \begin{pmatrix} f^-(q_1^n) \\ q_1^n \end{pmatrix}, \begin{pmatrix} f^+(q_1^n) \\ q_1^n \end{pmatrix}, \ldots, \begin{pmatrix} f^-(q_n^n) \\ q_n^n \end{pmatrix}, \begin{pmatrix} f^+(q_n^n) \\ q_n^n \end{pmatrix}) \tag{30}$$

$$Q'_n = (\begin{pmatrix} 1 - f^+(q_0^n) \\ q_0^n \end{pmatrix}, \begin{pmatrix} 1 - f^-(q_0^n) \\ q_0^n \end{pmatrix}, \ldots, \begin{pmatrix} 1 - f^+(q_n^n) \\ q_n^n \end{pmatrix}, \begin{pmatrix} 1 - f^-(q_n^n) \\ q_n^n \end{pmatrix}) \tag{31}$$

In particular, we directly obtain then the following observation:

$$Q_{n-1} \subseteq Q_n \quad \text{and} \quad Q'_{n-1} \subseteq Q_n \quad \text{for all } n > 0. \tag{32}$$

Next, for every i and n, we define the finite tests

$$S_i^n = Test_i(Q_n) \quad \text{and} \quad T_i^n = Test_i(Q'_n).$$

Then the second statement of Claim 24 implies that

$$S_i^{n-1} \preceq S_i^n,\ S_{i-1}^n \preceq S_i^n,\ T_i^{n-1} \preceq T_i^n,\ T_{i-1}^n \preceq T_i^n \text{ for all } i, n > 0.$$

Now, for all i, we can construct an infinite test S_i as follows: by (32), we can represent for every $n > 0$ the set $Cover(S_i^n) \setminus Cover(S_i^n)$ as a disjoint union of finitely many intervals (i.e., a *finite test*) X_i^n, wherein a list of intervals is computable in i and n because the same holds for S_i^n and S_i^{n-1}. So, we list the intervals in the finite sets $X_i^0, X_i^1, \dots$ in the infinite S_i. An infinite test T_i can be constructed likewise for every i.

As a next claim, we give an upper bound for the measure of the union of intervals in every constructed finite test that is implied by Claim 26.

▷ Claim 27. There is a constant c such that, for every naturals i and n, the measure of the union of all intervals from S_i^n and the measure of the union of all intervals from T_i^n are bounded from above by $2^{-i}c$.

By Claim 27, there exists a natural $m > 0$ such that, for every $n > 0$, the measures of $Cover(S_i^n)$ and $Cover(T_i^n)$ are both bounded from above by 2^{m-i}; therefore, by the compactness argument, the same holds for $Cover(S_i)$ and $Cover(T_i)$. Thus, the tests $(S_{m+1}, S_{m+2}, \dots)$ and $(T_{m+1}, T_{m+2}, \dots)$ are Martin-Löf tests, wherein, by the first statement of Claim 24, it holds that

$$Cover(S_{m+1}) \supseteq Cover(S_{m+2}) \supseteq \dots \quad \text{and} \quad Cover(T_{m+1}) \supseteq Cover(T_{m+2}) \supseteq \dots \tag{33}$$

Functions $\tilde{f}$ and f fulfill the weak Solovay inequality

Since β is Martin-Löf random, both Martin-Löf tests fail on it, hence there exist levels i_1 and i_2 such that $\beta \notin S_{i_1}$ and $\beta \notin T_{i_2}$. Therefore, by $\tilde{i} = \max\{i_1, i_2\} + 1$, it holds by (33) that

$$\beta \notin Cover(S_{\tilde{i}-1}) \cup Cover(T_{\tilde{i}-1}). \tag{34}$$

Next, we will successively prove two claims that the functions $\tilde{f}$ and f, respectively, fulfill the weak Solovay condition.

▷ Claim 28. For all rationals $q \in [0, \beta)$, it holds that $|\alpha - \tilde{f}(q)| < 2^{\tilde{i}}(\beta - x)$.

▷ Claim 29. For all reals $x \in [0, \beta)$, it holds that $|\alpha - f(q)| < 2^{\tilde{i}+1}(\beta - x)$.

5.3 The limit exists

At that moment, we already know from the previous paragraph that f fulfills weak Solovay inequality, i.e., the fraction $\frac{|\alpha - f(x)|}{\beta - x}$ is bounded on $[0, \beta)$.

▷ Claim 30. The limit point $\lim\limits_{x \nearrow \beta} \frac{\alpha - f(x)}{\beta - x}$ exists.

For a proof of this claim, see Appendix D.

5.4 The limit is unique

We prove the uniqueness of d for all weakly $\mathbb{R}$-translation functions from β to α by contradiction: suppose that there exist two translation functions f and g from β to α such that the (by the previous paragraph, existing) values $\lim\limits_{x\nearrow\beta}\frac{\alpha-f(x)}{\beta-x}$ and $\lim\limits_{q\nearrow\beta}\frac{\alpha-g(q)}{\beta-q}$ differ. By symmetry, without loss of generality, we can then pick rationals c and d such that

$$\lim_{q\nearrow\beta}\frac{\alpha-g(q)}{\beta-q}<c<d<\lim_{q\nearrow\beta}\frac{\alpha-f(q)}{\beta-q}. \tag{35}$$

By (35), for every rational $b<\beta$ that is close enough to β, it holds that

$$\frac{\alpha-g(q)}{\beta-q}<c \qquad \text{and} \qquad d<\frac{\alpha-f(q)}{\beta-q}.$$

Then, the real function $h=g-f$ is computable on $[b,\beta)$, fulfills

$$\lim_{x\nearrow\beta}h(x) = \lim_{x\nearrow\beta}g(x)-\lim_{x\nearrow\beta}f(x)=\alpha-\alpha=0 \text{ and} \tag{36}$$

$$\lim_{x\nearrow\beta}\frac{h(x)}{\beta-x} = \lim_{x\nearrow\beta}\frac{g(x)-f(x)}{\beta-x}=\lim_{x\nearrow\beta}\frac{\alpha-f(x)}{\beta-x}-\lim_{x\nearrow\beta}\frac{\alpha-g(x)}{\beta-x}=d-c>0. \tag{37}$$

From (36) and (37), we easily obtain the computability of β.

5.5 Case $d=0$

As a computable function, f is continuous on $[0,\beta)$ by [15, Theorem 4.3.1].

Therefore, if $d>0$ (the case $d<0$ is analogous), we can fix a constant $c<d$ and a rational $b<\beta$ such that $\frac{\alpha-f(x)}{\beta-x}\in[c,d]$ for every $x\in[b,\beta]$. The function

$$h(x)=\max\{f(y),f(b)+c(y-b):y\in[b,x]\}$$

is a strictly increasing $\mathbb{R}$-translation function with a (obviously) finite variation. Hence, its inverse f^{-1} defined on $[f(b),\alpha]$ is computable by [15, Inverse function Theorem] and has a finite variation on $[f(b),\alpha]$ (since f has a finite variation on $[b,\beta]$).

It also satisfies Solovay inequality (9) on some left neighborhood of α since

$$\lim_{y\nearrow\alpha}\frac{\beta-h(y)}{\alpha-y}=\lim_{x\nearrow\beta}\frac{\beta-x}{\alpha-h(x)}=\frac{1}{d},$$

hence, by Proposition 16, $\beta\leq^{\text{open}}_{\text{cL}}\alpha$.

Then α is Martin-Löf random because, by [14, Proposition 9(a)], the set of Martin-Löf reals is $\leq^{\text{open}}_{\text{cL}}$-closed upwards. ◀

6 Self-reducibility and Martin-Löf randomness

For a real α, a $\leq_{\text{S}}$-SELF-REDUCIBILITY of α is the reducibility $\alpha\leq_{\text{S}}\alpha$ witnessed by some $\mathbb{Q}$-translation function from α to itself. For other reducibilities discussed in this paper, the self-reducibility notion can be introduced likewise. The next characteristic of Martin-Löf randomness on the set on left-c.e. reals is implied by Theorem 5 applied for the Martin-Löf random left-c.e. reals $\alpha=\beta$.

▶ **Theorem 31.** *For every left-c.e. real α, the following two statements are equivalent:*

1. *α is Martin-Löf random;*

2. *Every weak* $\mathbb{R}$*-translation function* f *from* α *to itself with a finite total variation has the property* $\exists \lim\limits_{x \nearrow \alpha} \frac{\alpha - f(x)}{\alpha - x} = 1$.

Proof. First, we show the direction (1) $\implies$ (2). By Theorem 5 applied for $\alpha = \beta$, the limit value

$$d = \lim_{x \nearrow \alpha} \frac{\alpha - f(q)}{\alpha - q}$$

exists and does not depend on the choice of f witnessing $\alpha \leq_{\mathrm{cL}}^{\mathrm{open}} \alpha$, wherein, for $f = id$, we obviously have $d = 1$.

In order to prove $\neg$ (1) $\implies$ $\neg$ (2), we fix a Martin-Löf nonrandom left-c.e. real α and a Solovay test $S = ([l_n, r_n])_{n \in \mathbb{N}}$ that fails on α (i.e., $\alpha \in [l_n, r_n]$ for infinitely many r).

First, on the basis of S and $a_0, a_1, \dots$, we construct another Solovay test $T = (l'_n, r'_n)_{n \in \mathbb{N}}$ that fails on α and additionally fulfills

$$l'_0 < l'_1 < l'_2 < \dots < \alpha. \tag{38}$$

We do it by cutting the original test S in the following way: starting from $i_{-1} = t_{-1} = -1$, at every step $n \geq 0$, we denote with (i_n, t_n) the first index pair (i, t) such that

$$i > i_{n-1}, \quad t \notin \{t_0, t_1, \dots, t_{n-1}\}, \quad \text{and} \quad l_t < a_i$$

and put $[l'_n, r'_n] = [a_{i_n}, r_{t_n}]$ into the test T.

The constructed test T is obviously computable and has a finite measure since S has a finite measure and $[a_{i_n}, r_{t_n}] \subseteq [l_{t_n}, r_{t_n}]$; therefore, T is a Solovay test. Further, (38) holds since $l'_0, l'_1, \dots$ is a sub-sequence of $a_0, a_1, \dots$.

Finally, for every (of infinitely many) $\tilde{t}$ such that

$$\alpha \in [l_{\tilde{t}}, r_{\tilde{t}}],$$

we have $l_i < \alpha < r_{\tilde{t}}$ for every index i, hence there exists an index of a second order $\tilde{n}$ such that $\tilde{t} = t_{\tilde{n}}$ (in other words, the pair $(i_{\tilde{n}}, t_{\tilde{n}})$ where $t_{\tilde{n}} = \tilde{t}$ will be enumerated). Thus, by $a_{i_{\tilde{n}}} < \alpha$, we have $\alpha = [a_{i_{\tilde{n}}}, r_{t_{\tilde{n}}}]$. Therefore, the test T fails on α.

The following properties of the Solovay test T can be easily obtained from (38), its finite measure, and the existence of infinitely many indices i such that $\alpha \in [l'_i, r'_i]$:

$$l'_0, l'_1, \dots \nearrow \alpha \qquad \text{and} \qquad r'_0, r'_1, \dots \to \alpha.$$

Then, the function f defined by

$$f(x) = l'_n - (r'_n - l'_n) + \frac{x - l'_n}{l'_{n+1} - l'_n}\big(l'_{n+1} - (r'_{n+1} - l'_{n+1})\big) \text{ if } x \in [l'_n, l'_{n+1}] \text{ for some } n,$$

and $f(x) = l'_0$ on $[0, l'_0]$ is a piecewise linear function that fulfills $f(l'_i) = l'_i - (r'_i - l'_i)$ for all $i \geq 0$, hence its finite variation is equal to

$$\begin{aligned}
\sum_{i=0}^{n} |f(l'_{n+1}) - f(l'_n)| &= \sum_{i=0}^{n} \big((2l'_{n+1} - r'_{n+1}) - (2l'_n - r'_n)\big) \\
&\leq \sum_{i=0}^{n} (2l'_{n+1} - 2l'_n) + \sum_{i=0}^{n} (r'_n - l'_n) = 2(\alpha - l'_0) + \mu(T) < \infty,
\end{aligned}$$

where $\mu(T)$ denotes the (finite) measure of the Solovay test T.

Further, $f(l'_n) < l'_n < \alpha$ for all n, and $\lim_{x\nearrow\alpha} f(x) = \lim_{n\to\infty} \left(l'_n - (r'_n - l'_n)\right) = 2\lim_{n\to\infty} l'_n - \lim_{n\to\infty} r'_n = 2\alpha - \alpha = \alpha$, hence f is an $\mathbb{R}$-translation function from α to α, wherein it holds that

$$\limsup_{x\nearrow\alpha} \frac{\alpha - f(x)}{\alpha - x} \geq \limsup_{n\to\infty} \frac{(\alpha - l'_n) + (r'_n - l'_n)}{\alpha - l'_n} \geq 2,$$

since, for every n such that $\alpha \in [l'_n, r'_n]$, the corresponding fraction is larger that 2. ◀

▶ Remark 32. By [13, Proposition 53], speedability can be characterized on LEFT−CE (and even *defined* outside of LEFT−CE, see [13, Chapter 5] for further information) in terms of $\leq^{\mathrm{m}}_{\mathrm{S}}$-self-reducibility; in particular Theorem 5 directly implies that all (not only left-c.e.) Martin-Löf random reals are nonspeedable.

In the similar manner, Theorem 31 motivates to investigate an appropriate speedability notion that can be defined on $\mathbb{R}$ in terms of $\leq^{\mathrm{open}}_{\mathrm{cL}}$-self-reducibility. That can be the point of interest for a future research.

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A The proof of Proposition 7

Proof. We fix a computable enumeration $q_0, q_1, \dots$ of all rationals in $[0,1]$ by setting $q_0 = 0$, $q_1 = 1$, and, for every $n > 1$,

$$q_n = \{\frac{\tilde{m}}{\tilde{n}} :< \tilde{m}, \tilde{n} >= \min\big\{ < m, n >: \frac{m}{n} \notin \{q_0, \dots, q_{n-1}\}\big\}$$

and start the construction from defining a totally computable strictly increasing bijective function $g : \mathbb{Q}|_{[0,1]} \to \mathbb{Q}_2|_{[0,1]}$: set $g(q_0) = g(0) = 0$, $g(q_1) = g(1) = 1$ and, for every $n > 1$,

- sort the set $\{q_0, q_1, \dots, q_n\}$ increasingly and fix the indexes m_l and m_r of the left and right neighbors of q_n, respectively;
- let $g(q) = \frac{g(q_{m_l}) + g(q_{m_r})}{2}$.

We can easily see that g maps Cauchy sequences to Cauchy sequences, hence the function g continuously extensible. Therefore, by definition of $\mathbb{Q}$-translation function, we obtain for every real β that g is a strictly increasing $\mathbb{Q}$-translation function from β to $\lim\limits_{q_n \to \beta} g(q_n)$ — or, equivalently, for every real α that g is a strictly increasing $\mathbb{Q}$-translation from $\lim\limits_{q_n \to \alpha} g^{-1}(q_n)$ to α.

It remains to construct a d.c.e. real α such that $\beta := \lim\limits_{q_n \to \alpha} g^{-1}(q_n)$ is d.c.e. as well, and there are no nondecreasing $\mathbb{Q}$-translation functions from α to β.

A.1 Construction of sequences

We proceed the construction of two d.c.e. approximations $a_0, a_1, \dots \to \alpha$ and $b_0, b_1, \dots \to \beta$ in the standard finite injury way. Let $\phi_0, \phi_1, \dots$ be the enumeration of all partially computable function on rationals.

At the beginning of the construction, we set

$$a_{-1} := \frac{1}{2}, \quad I_0^0 := J_0^0 := [0,1], \quad \text{and} \quad d_0^0 = \frac{1}{4} \tag{39}$$

and, for every $\tilde{e} \geq 0$ such that $d_{\tilde{e}}^0$ is already defined, define first relevant interval with center in a_{-1} and radius $d_{\tilde{e}+1}^0$ by setting

$$< l_{\tilde{e}+1}^{-1}, r_{\tilde{e}+1}^{-1} >:= \min\left\{ < l, r >: \begin{cases} g(q_l) \in [a_{-1} - \frac{1}{4} d_{\tilde{e}+1}^0, a_{-1}], \\ g(q_r) \in [a_{-1}, a_{-1} + \frac{1}{4} d_{\tilde{e}+1}^0, a_{-1}], \\ q_r - q_l < \frac{1}{2} d_{\tilde{e}+1}^0 \end{cases} \right\}, \tag{40}$$

$$d_{\tilde{e}+1}^0 := [a_{-1} - d_{\tilde{e}+1}^0, a_{-1} + d_{\tilde{e}+1}^0], \tag{41}$$

$$I_{\tilde{e}+1}^0 := [a_{-1} - d_{\tilde{e}+1}^0, a_{-1} - d_{\tilde{e}+1}^0], \quad \text{and} \quad J_{\tilde{e}+1}^0 := [q_{r_{\tilde{e}+1}^{-1}} - q_{l_{\tilde{e}+1}^{-1}}] \tag{42}$$

and say that ϕ_e REQUIRES ATTENTION. We also set

$$a_{-1} := l_0^{-1} \quad \text{and} \quad w_{-1} := \lambda \text{ (empty word)}. \tag{43}$$

In the step $n \geq 0$, let (e, m, t) the minimal (relative to the Cantor tuple function $\pi_3(\cdot,\cdot,\cdot)$) triple such that

- ϕ_e requires attention,
- $q_m \notin \mathbb{Q}_2$,
- $\phi_e(q_m)[t] \downarrow$,
- $q_m \in I_e^n$.

Note then $q_m \neq g(\phi_e(q_m))$ since g is bijective, $q_m \notin \mathbb{Q}_2$, and $g(\phi_e(q_m)) \in \mathbb{Q}_2$. Accordingly, we define a_n in the interval I_e^n by setting

$$d_e^{n+1} := \begin{cases} \frac{1}{4}\max\{\phi_e(q_m), r(I_e^n)\} - q_m & \text{in case } q_m < g(\phi_e(q_m)), \\ \frac{1}{4}\min\{l(I_e^n), \phi_e(q_m)\} - q_m & \text{in case } q_m > g(\phi_e(q_m)), \end{cases} \tag{44}$$

$$a_n := \begin{cases} q_m + 2d_e^{n+1} & \text{in case } q_m < g(\phi_e(q_m)), \\ q_m - 2d_e^{n+1} & \text{in case } q_m > g(\phi_e(q_m)) \end{cases} \tag{45}$$

and, for every $\tilde{e} \geq e$, having $d_{\tilde{e}}^{n+1}$ already defined, define the next relevant interval $I_{\tilde{e}}^{n+1}$ with center in a_n and radius $d_{\tilde{e}}^{n+1}$ by setting

$$< l_{\tilde{e}+1}^n, r_{\tilde{e}+1}^n > := \min\left\{ < l, r >: \begin{cases} g(q_l) \in [a_n - \frac{1}{4}d_{\tilde{e}}^{n+1}, a_n], \\ g(q_r) \in [a_n, a_n + \frac{1}{4}d_{\tilde{e}}^{n+1}, a_n], \\ q_r - q_l < \frac{1}{2}d_{\tilde{e}}^{n+1}\}, \end{cases} \right\} \tag{46}$$

$$d_{\tilde{e}+1}^{n+1} := \min\{a_n - g(q_{l_{\tilde{e}+1}^n}), g(q_{r_{\tilde{e}+1}^n}) - a_n\}, \tag{47}$$

$$I_{\tilde{e}+1}^{n+1} := [a_n - d_{\tilde{e}+1}^{n+1}, a_n - d_{\tilde{e}+1}^{n+1}], \quad \text{and} \quad J_{\tilde{e}+1}^{n+1} := [q_{l_{\tilde{e}+1}^n}, q_{r_{\tilde{e}+1}^n}]. \tag{48}$$

Next, we set

$$e_n := e, \quad m_n := m, \quad t_n := t, \quad \text{and} \quad b_n = l_{e+1}^n \quad \text{as well as} \tag{49}$$

$$I_{\tilde{e}}^{n+1} := I_{\tilde{e}}^n \quad \text{and} \quad J_{\tilde{e}}^{n+1} := J_{\tilde{e}}^n \quad \text{for all } \tilde{e} < e \tag{50}$$

and, for the maximal index e' in the range $\{0, \ldots, e-1\}$ such that $\phi_{e'}$ remains rejected (if exists), let n' be the last step number when $\phi_{e'}$ has been rejected, i.e., $e' = e_{n'}$ (if there is no such index e', let $n' = -1$) and define

$$w_n = w_{n'}e_n.$$

At the end of the step n, we claim ϕ_{e_n} REJECTED and all $\phi_{\tilde{e}}$ for $\tilde{e} > e_n$ REQUIRING ATTENTION again.

A.2 Properties of the constructed sequences

First, for every n and e, the following embedding is directly implied by (47) and the monotonicity of g:

$$I_e^n \subseteq [g(l(J_e^n)), g(r(J_e^n))]. \tag{51}$$

Next, the function g is continuously extensible and striclty increasing, hence the sequence s_0, s_1 of computable reals defined by

$$s_n = \lim_{q \to a_n} g^{-1}(q) \tag{52}$$

converges to $\lim_{q \to \alpha} g^{-1}(q)$. At the beginning of the construction, we know that $d_0^0 = \frac{1}{4}$ and $|I_0^0| = 1$. Applying iteratively (40) through (42) to $\tilde{e}+1$ for all $\tilde{e}$ from 0 to infinity, we can obtain that

$$I_0^0 \supseteq I_1^0 \supseteq I_2^0 \supseteq \ldots \quad \text{and} \quad J_1^0 \supseteq J_2^0 \tag{53}$$

as well as, for every $e > 0$,

$$|I^0_{e+1}| \leq \frac{1}{4}|I^0_e| \quad \text{and} \quad |J^0_{e+1}| \leq \frac{1}{4}|I^0_e|; \tag{54}$$

$$a_0 \text{ is the center of } I^0_e \quad \text{and} \quad s_0 \in J^0_e \tag{55}$$

Similarly, for every $n \geq 0$, applying iteratively (46) through (48) for the step n to $\tilde{e}+1$ for all $\tilde{e}$ from e_n to infinity, we can obtain that

$$I^{n+1}_{e_n+1} \supseteq I^{n+1}_{e_n+2} \supseteq \dots \quad \text{and} \quad J^{n+1}_{e_n+1} \supseteq J^{n+1}_{e_n+2} \dots \tag{56}$$

as well as, for every $\tilde{e} \geq e_n + 1$,

$$|I^{n+1}_{\tilde{e}+1}| \leq \frac{1}{4}|I^{n+1}_{\tilde{e}}| \quad \text{and} \quad |J^{n+1}_{\tilde{e}+1}| \leq \frac{1}{4}|J^{n+1}_{\tilde{e}}|; \tag{57}$$

$$a_n \text{ is the center of } I^{n+1}_{\tilde{e}} \quad \text{and} \quad s_n \in J^{n+1}_{\tilde{e}}. \tag{58}$$

The next embedding property, due to its the nontrivial proof, has been selected as a separate claim.

▷ Claim 33. For every n, it holds that

$$I^{n+1}_{e_n+1} \subseteq I^n_{e_n} \quad \text{and} \quad I^{n+1}_{e_n+1} \subseteq J^n_{e_n} \tag{59}$$

with the upper bound for the length

$$|I^{n+1}_{e_n+1}| \leq \frac{1}{4}|I^n_{e_n}| \quad \text{and} \quad |J^{n+1}_{e_n+1}| \leq \frac{1}{4}|I^n_{e_n}|. \tag{60}$$

Proof. In the step n, we have

$$l(I^{n+1}_{e_n+1}) = a_n - d^{n+1}_{e_n+1} \geq a_n - g(q_{l^n_{e_n+1}}) \geq a_n - d^{n+1}_{e_n} \geq l(I^n_{e_n}), \tag{61}$$

$$r(I^{n+1}_{e_n+1}) = a_n + d^{n+1}_{e_n+1} \leq a_n + g(q_{l^n_{e_n+1}}) \leq a_n + d^{n+1}_{e_n} \leq r(I^n_{e_n}), \tag{62}$$

where the first inequalities in both lines hold by the left part of (48), second ones by (47), third ones by the first and second requirements in (46), respectively, and fourth by (44) and (45). In the similar way, we can obtain an upper bound for $|I^{n+1}_{e_n+1}|$:

$$|I^{n+1}_{e_n+1}| \leq 2d^{n+1}_{e_n+1} \leq \frac{1}{4} \cdot 2d^{n+1}_{e_n} \leq 2 \cdot \frac{1}{16}(r(I^n_{e_n}) - l(I^n_{e_n})) < \frac{1}{4}|I^n_{e_n}|. \tag{63}$$

For the intervals $J^{n+1}_{e_n+1}$ and $J^{n+1}_{e_n+1}$, we have

$$g(l(J^{n+1}_{e_n+1})) = g(q_{l^n_{e_n+1}}) \geq a_n - d^{n+1}_{e_n} \geq l(I^n_{e_n}) \geq g(l(J^n_{e_n})), \tag{64}$$

$$g(r(J^{n+1}_{e_n+1})) = g(q_{r^n_{e_n+1}}) \leq a_n + d^{n+1}_{e_n} \leq r(I^n_{e_n}) \leq g(r(J^n_{e_n})), \tag{65}$$

where the first inequalities hold by the third requirement in (46) and the monotonicity of g, the second ones in the same way as in (61) and (62), and the third one by (51). An upper bound for $|J^{n+1}_{e_n+1}|$ can be obtained as follows:

$$|J^{n+1}_{e_n+1}| < \frac{1}{2}d^{n+1}_{e_n} \leq \frac{1}{4}|I^n_{e_n}|. \tag{66}$$

◀

Note that, for every n, obtain from (58) that $a_n \in I^{n+1}_{e_n+1}$ and $s_n \in J^{n+1}_{e_n+1}$ and from $b_n = l^n_{e+1} = l(J^{n+1}_{e+1})$ that

$$b_n \in J^{n+1}_{e_n+1}. \tag{67}$$

Hence the latter claim implies that

$$a_n \in I^n_{e_n} \quad \text{and} \quad b_n, s_n \in J^n_{e_n}. \tag{68}$$

Now, the length of intervals $I^n_{e'}$ and $J^n_{e'}$ can be bounded explicitly.

▷ Claim 34. For every n and every $e' \geq e_n$, the length of intervals $I^n_{e'}$ and $J^n_{e'}$ have the following upper bound:

$$|I^n_e| \leq 4^{-e} \quad \text{and} \quad |J^n_e| \leq 4^{-e} \quad \text{for every } n \text{ and } e.$$

Proof. By induction on n: at the beginning of the construction, for $n = 0$, we have

$$|I^0_0| = |J^0_0| = 1 = 4^0$$

by (39) and, for all $e > 0$, consequent applying of (57) yields

$$|I^0_e| \leq 4^{-1}|I^0_{e-1}| \leq \dots 4^{-e}|I^0_0| \leq 4^{-e} \quad \text{and} \quad |J^0_e| \leq 4^{-1}|I^0_{e-1}| \leq \dots 4^{-e}|I^0_0| \leq 4^{-e}.$$

In the step n, having $|I^n_e| \leq 4^{-e}$ for all $e \geq n_e$ by inductional hypothesis, applying the inequality (60) yields

$$|J^{n+1}_{e_n+1}| \leq \frac{1}{4}|I^n_{e_n}| \leq \frac{1}{4}4^{-e} = 4^{-e-1}.$$

and, for all $e' \neq e_n$ consequent applying of (57) yields

$$|I^{n+1}_e| \leq \frac{1}{4}|I^{n+1}_{e-1}| \leq \dots \frac{1}{4^{e+e_n}}|I^{n+1}_{e_n+1}| \leq \frac{1}{4^{e-e_n-1}}4^{-e_n-1} = 4^{-e},$$

$$|J^{n+1}_e| \leq \frac{1}{4}|J^{n+1}_{e-1}| \leq \dots \frac{1}{4^{e+e_n}}|J^{n+1}_{e_n+1}| \leq \frac{1}{4^{e-e_n-1}}4^{-e_n-1} = 4^{-e}.$$

◀

▷ Claim 35. If $w_n = w_{n'}e_n$, then

$$a_{n'} \in I^n_{e_n} = I^{n'+1}_{e_n} \subseteq I^{n'+1}_{e_{n'}+1} \subseteq I^{n'}_{e_{n'}} \qquad \text{and} \qquad b_{n'}, s_{n'} \in J^n_{e_n} = J^{n'+1}_{e_n} \subseteq J^{n'+1}_{e'+1} \subseteq J^{n'}_{e_{n'}}.$$

Proof. First, we note that $a_n \in I^{n'+1}_{e_n}$ and $s_n \in J^{n'+1}_{e_n}$ by (58) and $b_n \in J^{n'+1}_{e_n}$ by (67).

The properties (59) applied for the step number n' yield

$$I^{n'+1}_{e_{n'}+1} \subseteq I^{n'}_{e_{n'}} \quad \text{and} \quad J^{n'+1}_{e_{n'}+1} \subseteq J^{n'}_{e_{n'}}. \tag{69}$$

Further, $w_n = w_{n'}e_n$ implies in particular that $e_{n'} < e_n$, hence we obtain from (56) for the step n' that

$$I^{n'+1}_{e_{n'}+1} \supseteq \dots \supseteq I^{n'+1}_{e_n-1} \supseteq I^{n'+1}_{e_n} \quad \text{and} \quad J^{n'+1}_{e_{n'}+1} \supseteq \dots \supseteq J^{n'+1}_{e_n-1} \supseteq J^{n'+1}_{e_n}. \tag{70}$$

Next, I_{e_n} and J_{e_n} remain unchanged in all steps in the range $n'+1, \dots, n-1$, hence

$$I^{n'+1}_{e_n} = I^n_{e_n} \quad \text{and} \quad J^{n'+1}_{e_n} = J^n_{e_n}. \tag{71}$$

Therefore, we obtain that

$$I^n_{e_n} = I^{n'+1}_{e_n} \subseteq I^{n'+1}_{e_{n'}+1} \subseteq I^{n'}_{e_{n'}} \quad \text{and} \quad J^n_{e_n} = J^{n'+1}_{e_n} \subseteq J^{n'+1}_{e_{n'}+1} \subseteq J^{n'}_{e_{n'}},$$

where, in both parts, the equality holds by (71), the first inequality by (70) and the second one by (69). ◀

The statement

$$\text{If } w_n \text{ contains } e_{n'}, \text{ then } I^n_{e_n} \subseteq I^{n'+1}_{e_{n'}+1} \subseteq I^{n'}_{e_{n'}} \text{ and } J^n_{e_n} \subseteq J^{n'+1}_{e_{n'}+1} \subseteq J^{n'}_{e_{n'}} \tag{72}$$

can be obtained by applying Claim 35 iteratively because in this case we have $w_n = w_{n'} e_{n'} \dots e_n$.

A.3 The limits α and β of sequences exist and are d.c.e.

We start the verification by proving that the sum of all $|a_n - a_{n-1}|$ is bounded. This fact would imply that the limit points $\alpha = \lim\limits_{n\to\infty} a_n$ and $\beta = \lim\limits_{n\to\infty} b_n$ exist and lie in DCE.

For this proof, we introduce an additional notation for the sum of interval between all consecutive points in some range:

$$s_{m,n} := \sum_{i=m+1}^{n} |a_i - a_{i-1}| \quad \text{and} \quad t_{m,n} := \sum_{i=m+1}^{n} |b_i - b_{i-1}|. \tag{73}$$

▷ Claim 36. Let $w_n = w_{n'} \dots e_n$ for some $n' < n$ and $e_{n'} < e_n$. Then, it holds that

$$s_{n',n} \le 4 \cdot 4^{-e_{n'}} \quad \text{and} \quad t_{n',n} \le 4 \cdot 4^{-e_{n'}}.$$

Proof. We prove the claim statement by induction on $n - n'$. Let $n_0, n_1, \dots, n_k$ be all such indexes $\tilde{n}$ that $w_{\tilde{n}} = w_{n'} e_{\tilde{n}}$. In particular, we it holds

$$n_0' = n' + 1, \quad w_n = w_{n'} e_{n_k} \dots e_n, \quad \text{and} \quad e_{n'} < e_{n_k} < \dots < e_{n_0} \tag{74}$$

Here, the inequality holds since, for every $i \in \{0, \dots, k\}$, $e_{n'}$ is the maximal index in the range $0,\dots,e_{n'} - 1$ of the function that remains rejected at the beginning for the step n_i. So, we can represent $s_{n',n}$ as a telescopic sum in the following way:

$$s_{n',n} = \underbrace{|a_{n_0} - a_{n'}|}_{\le 4^{-e_{n'}}} + \Big(\sum_{i=1}^{k} \underbrace{s_{n_i-1,n_{i-1}}}_{\le 4\cdot 4^{-e_{n_{i-1}}}}\Big) + \underbrace{s_{n_k,n}}_{\le 4\cdot 4^{-e_{n_k}}} + \sum_{i=1}^{k} \underbrace{|a_{n_i} - a_{n_i-1}|}_{\le 2\cdot 4^{-e_{n_i}}}$$

$$t_{n',n} = \overbrace{|b_{n_0} - b_{n'}|} + \Big(\sum_{i=1}^{k} \overbrace{t_{n_i-1,n_{i-1}}}\Big) + \overbrace{t_{n_k,n}} + \sum_{i=1}^{k} \overbrace{|a_{b_i} - b_{n_i-1}|}$$

Here, in both latter equalities, the upper bound for the first summands holds by Claim 34 since a_{n_0} lies in $I^n_{e_{n'}}$ and b_{n_0} lies in $I^n_{e_{n'}}$ by (58). The upper bounds for the second and third summands hold by inductional hypothesis since $a_{n_i-1} = w_{n'} e_{n_{i-1}} \dots e_{n_i-1}$ for every $i \in \{1, \dots, k\}$ by the choice of $n_0, \dots, n_k$ and $a_n = w_{n'} e_{n_k} \dots e_n$ by (74). It remains to argue the upper bounds for the fourth summands for every $i \in \{1, \dots, k\}$, i.e., to prove that

$$|a_{n_i} - a_{n_i-1}| < 2 \cdot 4^{e_{n_i}} \quad \text{and} |b_{n_i} - b_{n_i-1}| < 2 \cdot 4^{e_{n_i}} \tag{75}$$

By (68), we have

$$a_{n_i} \in I^{n_i}_{e_{n_i}}, \quad b_{n_i} \in J^{n_i}_{e_{n_i}}, \quad a_{n_{i-1}} \in I^{n_i}_{e_{n_{i-1}}}, \quad \text{and} \quad b_{n_i} \in J^{n_{i-1}}_{e_{n_{i-1}}}. \tag{76}$$

By Claim 35, we have

$$a_{n'} \in I^{n_i}_{e_{n_i}}, \quad a_{n'} \in I^{n_{i-1}}_{e_{n_{i-1}}}, \quad s_{n'} \in J^{n_i}_{e_{n_i}}, \quad \text{and} \quad s_{n'} \in J^{n_{i-1}}_{e_{n_{i-1}}} \tag{77}$$

By the choice of n_{i-1} and n_i, w_{n_i-1} contains $e_{n_{i-1}}$, hence we can apply (72) to the indices n_{i-1} and n_i and obtain

$$I^{n_{i-1}}_{e_{n_{i-1}}} \subseteq I^{n_i}_{e_{n_{i-1}}} \quad \text{and} \quad J^{n_{i-1}}_{e_{n_{i-1}}} \subseteq J^{n_i}_{e_{n_{i-1}}} \tag{78}$$

From the three latter observations, we obtain that

$$|a_{n_i} - a_{n_i-1}| \le |a_{n'} - a_{n_i}| + |a_{n'} - a_{n_i-1}| \le |I^{n_i}_{e_{n_i}}| + |I^{n_{i-1}}_{e_{n_{i-1}}}| \le 4^{-e_{n_i}} + 4^{-e_{n_{i-1}}},$$

$$|b_{n_i} - b_{n_i-1}| \le |s_{n'} - b_{n_i}| + |s_{n'} - b_{n_i-1}| \le |J^{n_i}_{e_{n_i}}| + |J^{n_{i-1}}_{e_{n_{i-1}}}| \le 4^{-e_{n_i}} + 4^{-e_{n_{i-1}}},$$

where, in both lines, the last inequalities hold by Claim 34. Finally, we know from the right part of (74) that $e_{n_{i-1}} > e_{n_i}$, hence $4^{-e_{n_i}} + 4^{-e_{n_{i-1}}} \le 2 \cdot 4^{-e_{n_i}}$. That concludes the proof of (75).

Summarizing all upper bounds, we obtain that

$$\begin{aligned} s(n', n) \le& 4^{-e_{n'}} + 4\sum_{i=0}^{k-1} 4^{-e_{n_i}} + 4 \cdot 4^{-e_{n_k}} + 2 \cdot \sum_{i=1}^{k} 4^{-e_{n_i}} \\ \le& 4^{-e_{n'}} + 9\sum_{i=0}^{k} 4^{-e_{n_i}} \le 4^{-e_{n'}} + 9 \cdot \frac{1}{3} 4^{-e_{n'}} = 4^{-e_{n'}}, \end{aligned}$$

and the same for $t_{n',n}$. Here, all the inequalities are straightforward except the third one; the latter follows from

$$\sum_{i=0}^{k} 4^{-e_{n_k}} \le \sum_{i=0}^{\infty} 4^{-e_{n'}+i+1} = 4^{-e_{n'}}(4^{-1} + 4^{-2} + \dots) = \frac{1}{3} \cdot 4^{-e_{n'}},$$

where the inequality is implied by the right part of (74). ◀

Now, we can finish the proof that $\sum_{n=1}^{\infty} |a_n - a_{n-1}|$ and $\sum_{n=1}^{\infty} |b_n - b_{n-1}|$ are finite.

The following discussion is standard for all construction of finite injury type: in case ϕ_e has been rejected in some stage n (i.e., $e_n = e$), that either ϕ_e will never require attention again (and therefore, will never be rejected once more) or ϕ_e will require attention again after the step $n' > n$ on that some function $\phi_{e'}$ for $e' < e$ will be rejected. For every e, there are only e indexes in the range $0, \dots, e-1$, hence the function ϕ_e can be rejected only $2^e - 1 < \infty$ times.

Therefore, we can fix a noncomputable index sequence $k_0 < k_1 < \dots$ that contains all such indexes k such that ϕ_k will be rejected at some step and never requires attention again. Now, we consider the index sequence $n_0, n_1, \dots$ of TRUE STAGES defined as follows: for every i, let n_i be the last step on that the function ϕ_{k_i} has been rejected, i.e., such that $e_{n_i} = k_i$. In particular, the last definition implies by construction that

$$n_0 < n_1 < \dots \quad \text{and} \quad e_{n_0} < e_{n_1} < \dots, \tag{79}$$

whence, for every i, we straightforwardly obtain by definition of final stage n_i that $e_{\tilde{n}} > e_{n_i}$ for every $\tilde{n} > n_i$. Moreover, for a fixed i, the existence of a step $\tilde{n} \in \{n_i + 1, \dots, n_{i+1} - 1\}$ such that $e_{\tilde{n}} \le e_{n_{i+1}}$ would lead to contradiction since, in that case, we consider such $\tilde{n}$ with the minimal $e_{\tilde{n}}$. Since $\tilde{n}$ is not a final stage, there exist a step $n' > \tilde{n}$ requires that $e_{\tilde{n}}$ requires attention again. Thus, $e_{n'} < e_{\tilde{n}} (\le e_{n_{i+1}})$, hence the definition of a final stage implies that $n' < n_{i+1}$. But the inequalities $e_{n'} < e_{n_{i+1}}$ and $n' < n_{i+1}$ together contradict the minimality of $\tilde{n}$.

By the latter discussion, we obtain for every i the following property:

$$e_{\tilde{n}} < e_{n_{i+1}} \text{ for every } \tilde{n} > e_{n_{i+1}} \text{ except } \tilde{n} = n_{i+1}. \tag{80}$$

In particular, the latter property implies that, for a fixed i, none of the functions $\phi_0, \phi_1, \dots, \phi_{n_i}$ will be rejected in any step $\tilde{n} > n_i$, hence we can obtain by an obvious induction of $\tilde{n}$ that

$$w_{\tilde{n}} \text{ contains } n_i \text{ for every } \tilde{n} > n_i. \tag{81}$$

Thus, by (68) and (72), we have

$$a_n \in I^{n_i}_{e_{n_i}} \quad \text{and} \quad b_n \in J^{n_i}_{e_{n_i}} \quad \text{for all } n > n_i \text{ for every } i, \tag{82}$$

hence

$$\sum_{n=1}^{\infty} |a_n - a_{n-1}| = \quad s_{0e_0} + \sum_{i=1}^{\infty} s_{e_n e_{n-1}} \leq s_{0e_0} + \sum_{i=1}^{\infty} 4 \cdot 4^{-e_{n_{i-1}}} < s_{0e_0} + 4 \cdot \frac{4}{3} < \infty,$$

$$\sum_{n=1}^{\infty} |b_n - b_{n-1}| = \quad t_{0e_0} + \sum_{i=1}^{\infty} t_{e_n e_{n-1}} \leq t_{0e_0} + \sum_{i=1}^{\infty} 4 \cdot 4^{-e_{n_{i-1}}} < s_{0e_0} + 4 \cdot \frac{4}{3} < \infty.$$

where, in both lines, the first inequality holds by Claim 34 since $e_{n_0} < e_{n_1} < \dots$ and the second one is implied by $4^0 + 4^{-1} + 4^{-2} + \dots = \frac{4}{3}$.

In particular, the latter discussion implies that the sequences $a_0, a_1, \dots$ and $b_0, b_1, \dots$ converge. We denote their limits with α and β, respectively.

A.4 The reals α and β satisfy the proposition assertion

First, we show that

$$\beta = \lim_{q \to \alpha} g^{-1}(q). \tag{83}$$

Since, for every $i \in \mathbb{N}$ and all $n \geq n_i$, the points s_n and b_n lie in the interval $J^{n_i}_{e_{n_i}}$ by (68), where $|J^{n_i}_{e_{n_i}}| \leq 4^{-e_{n_i}} = 4^{-k_i} \leq 4^{-i+1} \underset{i \to \infty}{\to} 0$, the sequence $s_0 - b_0, s_1 - b_1, \dots$ tends to 0, therefore $\beta = \lim_{n \to \infty} b_n = \lim_{n \to \infty} s_n$. Since g^{-1} is continuously extensible, we obtain from definition of s_n that

$$\beta = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \big(\lim_{q \to a_n} g^{-1}(q) \big) = \lim_{q \to \alpha} g^{-1}(q),$$

that concludes the proof of (83).

▷ Claim 37. for every true stage n_i, $\phi_{e_{n_i}}$ is not a translation function from α to $\lim_{q_n \to \alpha} g^{-1}(q_n)$.

Proof. Let n_i be a true stage. We know by (81) that $w_{\tilde{n}}$ contains n_i. Therefore, by (68) and Claim 35,

$$a_{\tilde{n}} \in I^{\tilde{n}}_{e_{\tilde{n}}} \subseteq I^{n_i+1}_{e_{n_i}+1} \quad \text{for every } \tilde{n} > n_i. \tag{84}$$

Since the interval $I^{n_i+1}_{e_{n_i}+1}$ is compact and $\alpha = \lim_{n \to \infty} a_n$, we obtain from (84) that

$$\alpha \in I^{n_i+1}_{e_{n_i}+1}. \tag{85}$$

On the other hand, the interval $I^{n_i+1}_{e_{n_i}+1}$ is contained between the points $q_{m_{n_i}}$ and $g(\phi_{e_{n_i}}(q_{m_{n_i}}))$ by choice of m_{n_i} and e_{n_i}, hence we obtain the case distinction

$$1)\ q_{m_{n_i}} < \alpha < g(\phi_{e_{n_i}}(q_{m_{n_i}})) \qquad \text{and} \qquad 2)\ g(\phi_{e_{n_i}}(q_{m_{n_i}})) < \alpha < q_{m_{n_i}}.$$

We rename $q := q_{m_{n_i}}$ and $\phi := \phi_{e_{n_i}}$ and consider these cases separately.

Case 1: $q < \alpha < g(\phi(q))$.

Then we have $\phi(q) \in RC(\alpha)$. Since g is a nondecreasing $\mathbb{Q}$-translation function from β to α, we directly obtain that $\phi(q) \in RC(\beta)$. Hence $\phi(= \phi_{e_{n_i}})$ cannot be a $\mathbb{Q}$-translation function from α to β since it maps $q \in LC(\beta)$ to $\phi(q) \in RC(\alpha)$, which contradicts Definition 6.

Case 2: $g(\phi(q)) < \alpha < q$.

Then we have $\phi(q) \in LC(\alpha)$. Since g is a $\mathbb{Q}$-translation function from β to α, we obtain by Definition 6 that $\phi(q) \in LC(\beta)$. Hence $\phi(= \phi_{e_{n_i}})$ cannot be a nondecreasing $\mathbb{Q}$-translation function from α to β since it maps $q \in RC(\beta)$ to $\phi(q) \in LC(\alpha)$, which cannot be true because a monotone translation function cannot map points on $RC(\beta)$ left from α. ◀

Thus, for some index e, if the function ϕ_e has been rejected at some step n and never requires attention again, than $n = n_e$ is a true stage, hence the latter claim implies that ϕ_e cannot be a nondecreasing $\mathbb{Q}$-translation function from α to β.

Otherwise, there exist no true stage n such that $e_n = e$. That means by (79) that there exists a true stage index k such that $e_{n_k} < e < e_{n_{k+1}}$, and the function ϕ_e will requires attention in every step $n' \geq n_k + 1$. Fixing this k, we obtain that

$$\alpha \in I^{n_{k+1}}_{e_{n_{k+1}}} \subseteq I^{n_k+1}_{e_{n_{k+1}}} \subseteq I^{n_k+1}_{e} = I^{\tilde{n}}_{e} \text{ for every } \tilde{n} \geq n_k + 1.$$

Here, the first interval contains α by (85), the first inclusion holds by Claim 35, and the second one by (59). Finally, the equality follows from (80) since, on every step $\tilde{n}$ in the range $n_k + 1, \ldots, \tilde{n}$, the interval I_e remains unchanged.

Thus, there exists a nondyadic rational $q_m \in I^{n_k+1}_{e} \cap [0, \alpha)$, such that

$$m > \pi_3(m_{n_{k+1}}, e_{n_{k+1}}, t_{n_{k+1}}).$$

If $\phi_e(q_m)[t] \downarrow$ for some t, then we obviously have

$$\pi_3(m, e, t) > m > \pi_3(m_{n_{k+1}}, e_{n_{k+1}}, t_{n_{k+1}}),$$

hence the triple (m, e, t) will fulfill all requirements of some step $n' > n_{k+1}$, which implies in particular that $e_{n'} = e < e_{n_{k+1}}$, hence $e_{n_{k+1}}$ will require attention again after the step n', and thus n_{k+1} is not a true stage, a contradiction.

Therefore, the function ϕ_e is undefined in $q_m \in LC(\alpha)$, and thus it cannot be a $\mathbb{Q}$-translation function from α to β.

We have proven that no function in the enumeration $\phi_0, \phi_1, \ldots$, which contains all partially computable functions from rationals to rationals, can be a nondecreasing $\mathbb{Q}$-translation function from α to β, that concludes the proof of the proposition.

◀

B The proof of Lemma 19

Proof. Let X be a countable dense subset of I, and let $x_0, x_1, \ldots$ be an enumeration of X. By definition of V_f, we can fix for every $i \in \mathbb{N}$ a family of finite sets $\tilde{X}_n = (\tilde{x}^n_0, \ldots, \tilde{x}^n_{\ell_n})$ (not necessarily lying in X) such that

$$\tilde{x}^n_0 < \tilde{x}^n_1 < \cdots < \tilde{x}^n_{\ell_n} \text{ for all } n, \tag{86}$$

$$v_n := \sum_{i=0}^{\ell_n - 1} \left| f(\tilde{x}^n_{i+1}) - f(\tilde{x}^n_i) \right| \geq V_f - 2^{-n} \text{ for all } n, \text{ and} \tag{87}$$

$$\tilde{X}_0 \subseteq \tilde{X}_1 \subseteq \ldots \text{without loss of generality.} \tag{88}$$

For every $n \geq 0$, let $x^n_0, \ldots, x^n_{k_n}$ and $y^n_0, \ldots, y^n_{k_n}$ be defined as in the lemma statement and set

$$s_n := \sum_{i=0}^{k_n - 1} |y^n_{i+1} - y^n_i|.$$

Then, the sequence $s_0, s_1, \dots$ is monotone nondecreasing, and, for every n, the definition of V_f as a supremum implies that $V_f \ge s_n$, hence we obtain the inequality $\limsup_{n\to\infty} s_n \le V_f$. So, for obtaining (13), it remains to prove that $\liminf_{n\to\infty} s_n \ge V_f$.

The latter property can be demonstrated as follows. First, the continuity of f implies for every n and every $j \in \{0, \dots, \ell_n\}$ the existence of an interval

$$I_j^n := [\tilde{x}_j^n - \varepsilon_j^n, \tilde{x}_j^n + \varepsilon_j^n] \text{ such that } \left|f(x) - f(\tilde{x}_j^n)\right| \le 2^{-(n+\ell_n+1)} \text{ for all } x \in I_j^n. \tag{89}$$

Fix $n > 0$. Since X is dense on I, there exists an index $k_n (\ge k_{n-1}$ if $n > 0)$ of its enumeration $x_0, x_1, \dots$ such that

for every $j \in \{0, \dots, \ell_n\}$ there exists an index $m_j \le k_n$ such that $x_{m_j} \in I_j^n$,

and we have $x_{m_0} < x_{m_1} < \dots < x_{m_{\ell_n}}$,

so we can define $y_i^n = x_{m_i}$ for all $i \in \{0, \dots, k_n\}$ and obtain

$$\begin{aligned}
s_n &= \sum_{i=0}^{k_n-1} \left|f(y_{i+1}^n) - f(y_i^n)\right| \ge \sum_{j=0}^{\ell_n-1} \left|f(y_{j+1}^n) - f(y_j^n)\right| \\
&\ge \sum_{j=0}^{\ell_n-1} \left(-\left|f(y_{j+1}^n) - f(\tilde{x}_{j+1}^n)\right| + \left|f(\tilde{x}_{j+1}^n) - f(\tilde{x}_j^n)\right| - \left|f(\tilde{x}_j^n) - f(y_j^n)\right|\right) \\
&\ge \sum_{j=0}^{\ell_n-1} \left|f(\tilde{x}_{j+1}^n) - f(\tilde{x}_j^n)\right| - 2\sum_{j=0}^{\ell_n} \left|f(y_j^n) - f(\tilde{x}_j^n)\right| \ge v_n - 2 \cdot (\ell_n + 1) \cdot 2^{-(n-\ell_n+1)} \\
&\ge (V_f - 2^{-n}) - (\ell_n + 1) \cdot 2^{-(n+\ell_n)} \ge V_f - 2^{-n+1},
\end{aligned}$$

which implies that $\liminf_{n\to\infty} s_n \ge V_f$. Here, the first inequality holds since all indexes $m_0, m_1, \dots, m_{\ell_n}$ are different and lie in $\{0, 1, \dots, k_n\}$ by choice of k_n, the second follows from the subadditivity of the absolute value operator, the fourth is implied by definition of v_n and putting $x := y_j^n$ for all $j \in \{0, \dots, \ell_j\}$ in the inequality (89), and the fifth inequality holds by (87). ◀

C The proofs of claims in Paragraph 5.2

C.1 The proof of Claim 22

Since the enumeration $q_0, q_1, \dots$ contains the set $\mathbb{Q}|_I$ which is a dense subset of I, we obtain by Lemma 19 that

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} |f(q_{i+1}^n) - f(q_i^n)| = M, \tag{90}$$

Further, from the left part of 18 we obtain that

$$\sum_{n\in\mathbb{N}} |\tilde{f}(q_n) - f(q_n)| \le \sum_{n\in\mathbb{N}} 2^{-2n} = \frac{4}{3}, \tag{91}$$

which implies for every tuple $(q_0^n, \dots q_n^n)$ defined as in the claim condition that

$$\sum_{i=0}^{n-1} |\tilde{f}(q_{i+1}^n) - \tilde{f}(q_i^n)| \leq \sum_{i=0}^{n-1} \left(|\tilde{f}(q_{i+1}^n) - f(q_{i+1}^n)| + |f(q_{i+1}^n) - f(q_i^n)| + |f(q_i^n) - \tilde{f}(q_i^n)|\right)$$

$$\leq \sum_{i=0}^{n-1} |f(q_{i+1}^n) - f(q_i^n)| + 2\sum_{i=0}^{n} |\tilde{f}(q_i^n) - f(q_i^n)|$$

$$\leq \sum_{i=0}^{n-1} |f(q_{i+1}^n) - f(q_i^n)| + 2\sum_{n\in\mathbb{N}} |\tilde{f}(q_n) - f(q_n)|$$

$$= \sum_{i=0}^{n-1} |f(q_{i+1}^n) - f(q_i^n)| + 2\cdot\frac{4}{3} < M + \frac{8}{3},$$

where the third inequality holds by (90).

C.2 The proof of Claim 23

We prove the claim assertion by contradiction: supposing that there exists a constant $c = 2^{-j}$ where $j > 0$ such that $\limsup_{q\nearrow\beta} |\alpha - \tilde{f}(q)| > 2^{-j}$, we obtain by density of the sequence $q_0, q_1, \dots$ on the interval $[0, \beta)$ the existence of an index sequence $n_0, n_1, \dots$ such that

$$q_{n_i} \nearrow \beta \quad \text{and} \quad |\alpha - \tilde{f}(q_{n_i})| \geq 2^{-j}. \tag{92}$$

As a weakly $\mathbb{R}$-translation function, f fulfills (7), hence there exists a rational $b < \beta$ such that

$$|\alpha - f(x)| < 2^{-(j+1)} \text{ for every } x \in [b, \beta). \tag{93}$$

Due to $q_{n_i} \nearrow \beta$, there is an index $\tilde{i} > j$ such that $q_{n_{\tilde{i}}} \in [b, \beta)$. Then we obtain a contradiction to (92) by

$$|\alpha - \tilde{f}(q_{n_{\tilde{i}}})| < \underbrace{|\alpha - f(q_{n_{\tilde{i}}})|}_{<2^{-(j+1)} \text{ by (93)}} + \underbrace{|f(q_{n_{\tilde{i}}}) - \tilde{f}(q_{n_{\tilde{i}}})|}_{\leq 2^{-2n_{\tilde{i}}}} < 2^{-j}.$$

Here the last inequality follows from $n_{\tilde{i}} \geq \tilde{i}$ (which is true since the index sequence $n_0, n_1, \dots$ is strictly increasing) and $2\tilde{i} \geq j + 1$ (which holds by choice of $\tilde{i}$ and j).

The proof of Claim 24

Let $P := (\binom{a_0}{b_0}, \binom{a_1}{b_1}, \dots, \binom{a_n}{b_n})$ and Q be two test sets where $P \subseteq Q$, and let i, j be two naturals where $i < j$.

Then the finite test $Test_i(P)$ is a subset of the finite test $Test_i(Q)$ since every intersection of an interval $I[k,m] = R[\binom{a_k}{b_k}, \binom{a_m}{b_m}] = [b_m, b_k + \frac{a_m - a_k}{2^{i+1}}]$ where $0 \leq k < m \leq n$ with $[0,1)$ added into the test $Test_i(P)$ will also be added into the test $Test_i(Q)$ because Q also contains the pairs $\binom{a_k}{b_k}$ and $\binom{a_m}{b_m}$. Hence, we directly obtain

$$\bigcup_{\binom{p}{q}, \binom{p'}{q'} \in P} R[\binom{p}{q}, \binom{p'}{q'}] \subseteq \bigcup_{\binom{p}{q}, \binom{p'}{q'} \in Q} R[\binom{p}{q}, \binom{p'}{q'}];$$

and thus, $Test_i(P) \preceq Test_i(Q)$.

Further, for every two indexes k and m where $0 \leq k < m \leq n$, it holds that

$$I^j[k,m] = [b_m, b_k + \frac{a_m - a_k}{2^{j+1}}] \subseteq [b_m, b_k + \frac{a_m - a_k}{2^{i+1}}] = I^i[k,m],$$

hence we obtain

$$\bigcup_{0 \leq k < m \leq n} I^j[k,m] \subseteq \bigcup_{0 \leq k < m \leq n} I^i[k,m];$$

and thus, $Test_j(P) \preceq Test_i(P)$.

C.3 The proof of Claim 25

Fix k and l such that $0 \leq k < l \leq n$.

In order to prove the first implication, assume that k, l fulfill $\frac{a_l - a_k}{b_l - b_k} \leq 2^{i+1}$, which is equivalent to

$$b_l - b_k \geq \frac{a_l - a_k}{2^{i+1}}, \tag{94}$$

and fix $m > l$. For every real $x \in I[k,m]$, definition of $I[k,m]$ implies the inequality $b_m \leq x \leq b_k + \frac{a_m - a_k}{2^{i+1}}$. Inter alia, it means that

$$b_l < b_m \leq x \qquad \text{and} \qquad x - b_k \leq \frac{a_m - a_k}{2^{i+1}}. \tag{95}$$

Hence, we obtain that

$$x - b_l = (x - b_k) - (b_l - b_k) \leq \frac{a_m - a_k}{2^{i+1}} - \frac{a_l - a_k}{2^{i+1}} = \frac{a_m - a_l}{2^{i+1}}, \tag{96}$$

where the inequality follows from the right part of (95) and (94). The right side of (23) is implied by the left part of (95) and (96).

For the second implication, assume that k, l fulfill $\frac{a_l - a_k}{b_l - b_k} \geq 2^{i+1}$, which is equivalent to

$$b_l - b_k \leq \frac{a_l - a_k}{2^{i+1}}. \tag{97}$$

In case $m \leq l$, the right side of (24) is obvious since $I[l,m] = \emptyset$, so it suffices to consider $m > l$. For every real $x \in I[l,m]$, definition of $I[l,m]$ implies the inequality $b_m \leq x \leq b_l + \frac{a_m - a_l}{2^{i+1}}$. Inter alia, it means that

$$b_k < b_l < b_m \leq x \qquad \text{and} \qquad x - b_l \leq \frac{a_m - a_l}{2^{i+1}}. \tag{98}$$

Hence, we obtain similar as in the proof of previous implication that

$$x - b_k = (x - b_l) + (b_l - b_k) \leq \frac{a_m - q_l}{2^{i+1}} + \frac{a_l - a_k}{2^{i+1}} = \frac{a_m - a_k}{2^{i+1}}. \tag{99}$$

The right part of (24) is implied by the left part of (98) and (99).

C.4 The proof of Claim 26

First, fixing some $j \in \{0,\dots,s\}$, we show the somewhat technical inclusion for the intervals of a form $I[i_j,\cdot]$

$$\forall k < i_{j+1}\Big(I[k,i_j]=\emptyset \ \text{ and } \ \forall m > i_j\big(I[k,m] \subseteq I[i_j,m]\big)\Big). \tag{100}$$

by case distinction for k.

- In case $k = i_h$ for some $h < j$, it holds that

$$\frac{a_{i_j}-a_{i_h}}{b_{i_j}-b_{i_h}} = \frac{(a_{i_j}-a_{i_{j-1}})+\cdots+(a_{i_{h+1}}-a_{i_h})}{(b_{i_j}-b_{i_{j-1}})+\cdots+(b_{i_{h+1}}-b_{i_h})} \le 2^{i+1}, \tag{101}$$

  where the inequality follows from (26) applied for indices $h,\dots,j-1$ since, for all natural l and all reals $A_1,\dots,A_l \ge 0$ and $B_1,\dots,B_l,C>0$, the l equalities $\frac{A_1}{B_1}\le C,\dots,\frac{A_l}{B_l}\le C$ imply together that $\frac{A_1+\cdots+A_l}{B_1+\cdots+B_l}\le C$.
  Therefore, we obtain by (21) that $I[i_h,i_j]=\emptyset$, and (23) implies for every $m > i_j$ that

$$I[i_h,m] \subseteq I[i_j,m]. \tag{102}$$

- In case $k \in \{i_h+1,\dots,i_{h+1}-1\}$ for some $h<j$, it holds by choice of i_h that

$$\frac{a_k-a_{i_h}}{b_k-b_{i_h}} > 2^{i+1}. \tag{103}$$

  First, we obtain that $I[i_h,i_j]=\emptyset$ by (21) since

$$\frac{a_{i_j}-a_k}{b_{i_j}-b_k} = \frac{(a_{i_j}-a_{i_h})-(a_k-a_{i_h})}{(b_{i_j}-b_{i_h})-(b_k-b_{i_h})} \le 2^{i+1}, \tag{104}$$

  where the inequality follows from (101) and (103) because, for all reals $A_1,A_2\ge 0$ and $B_1,B_2,C>0$, the two equalities $\frac{A_1}{B_1}\le C$ and $\frac{A_2}{B_2}>C$ imply together that $\frac{A_1-A_2}{B_1-B_2}\le C$.
  Second, we obtain for every $m>i_j$ that

$$I[k,m] \subseteq I[i_h,m] \subseteq I[i_j,m], \tag{105}$$

  where the left side is implied by (24) due to (104), and the right side holds by (102).
- In case $k \in \{i_j,\dots,i_{j+1}-1\}$, we straightforwardly obtain from $k \ge i_j$ that $I[k,i_j]=\emptyset$. For $k=i_j$, the right side of (100) is trivial; for $k>i_j$, we obtain from the choice of i_j that

$$\frac{a_k-a_{i_j}}{b_k-b_{i_j}} > 2^{i+1}, \tag{106}$$

  and thus (24) implies for every $m>i_j$ that

$$I[k,m] \subseteq I[i_j,m]. \tag{107}$$

Now, using (100), we can proof the claim assertion. Let $x \notin [b_{i_j}, b_{i_j}+\frac{a_{h_j}-a_{i_j}}{2^{i+1}}]$ for every j in the range. We prove the claim by contradiction: assuming that there exists an index pair (k,m) such that $x \in I[k,m]$ (in particular, it implies that $k<m$, we fix the index $\tilde{j} \in \{0,\dots,s\}$ such that

$$i_{\tilde{j}} \le m < i_{\tilde{j}+1}. \tag{108}$$

The indices $\tilde{j}$, k and m fulfill the conditions of (100) since $k < l < i_{\tilde{i}+1}$ and $m \geq \tilde{i}$; thus, by (100), we obtain that

$$x \in I[k, m] \subseteq I[i_{\tilde{j}}, m]. \tag{109}$$

Next, by choice (108) of $\tilde{j}$ and definition (28) of $h_{\tilde{j}}$, it holds that $a_{h_{\tilde{j}}} \geq a_m$, which implies that

$$b_{i_{\tilde{j}}} + \frac{a_m - a_{i_{\tilde{j}}}}{2^{i+1}} \leq b_{i_{\tilde{j}}} + \frac{a_{h_{\tilde{j}}} - a_{i_{\tilde{j}}}}{2^{i+1}}. \tag{110}$$

On the other hand, we know by (108) that $i_{\tilde{j}} \leq m$, hence we obtain by $b_0 \leq b_1 \leq \cdots \leq b_n$ that $b_m \leq b_{i_{\tilde{j}}}$. Together with (110), it implies that

$$I[i_{\tilde{j}}, m] = [b_m, b_{i_{\tilde{j}}} + \frac{a_m - a_{i_{\tilde{j}}}}{2^{i+1}}] \subseteq [b_{i_{\tilde{j}}}, b_{i_{\tilde{j}}} + \frac{a_{h_{\tilde{j}}} - a_{i_{\tilde{j}}}}{2^{i+1}}], \tag{111}$$

where the first equality holds by (21) since $x \in I[i_{\tilde{j}}, m] \neq \emptyset$ by (109).

Finally, from (109) and (111), we obtain

$$x \in [b_{i_{\tilde{j}}}, b_{i_{\tilde{j}}} + \frac{a_{h_{\tilde{j}}} - a_{i_{\tilde{j}}}}{2^{i+1}}],$$

which contradicts to the claim condition.

The proof of Claim 27

For Q_n defined as in (30), the property (29) turns into

$$Cover(S_i^n) = Cover(Test_i(Q_n)) \subseteq \bigcup_{j \in \{0,\ldots,s\}} [q_{i_j}^n, q_{i_j}^n + \frac{f^+(q_{h_j}^n) - f^-(q_{i_j}^n)}{2^{i+1}}]$$

for appropriate indices $i_0, \ldots, i_s$ where (sic!) $0 \leq i_0 \leq i_1 \cdots \leq i_n \leq n$, hence we obtain the following upper bound for the Lebesgue measure of the union of intervals in $Cover(S_i^n)$:

$$\mu(Cover(Test(Q_i))) \leq \sum_{j \in \{0,\ldots,s\}} \mu\big([q_{i_j}^n, q_{i_j}^n + \frac{f^+(q_{h_j}^n) - f^-(q_{i_j}^n)}{2^{i+1}}]\big)$$
$$= 2^{-(i+1)} \sum_{j \in \{0,\ldots,s\}} (f^+(q_{h_j}^n) - f^-(q_{i_j}^n))$$
$$= 2^{-(i+1)} \sum_{j \in \{0,\ldots,s\}} \Big(\big(f^+(q_{h_j}^n) - \tilde{f}(q_{h_j}^n)\big) + \big|\tilde{f}(q_{h_j}^n) - \tilde{f}(q_{i_j}^n)\big| + \big(\tilde{f}(q_{i_j}^n) - f^-(q_{i_j}^n)\big)\Big)$$
$$\leq 2^{-(i+1)} \Bigg(\sum_{j \in \{0,\ldots,s\}} \big|\tilde{f}(q_{h_j}^n) - \tilde{f}(q_{i_j}^n)\big|$$
$$+ \sum_{j \in \{0,\ldots,s\}} \big(f^+(q_{h_j}^n) - f^-(q_{h_j}^n)\big) + \sum_{j \in \{0,\ldots,s\}} \big(f^+(q_{i_j}^n) - f^-(q_{i_j}^n)\big)\Bigg)$$
$$\leq 2^{-(i+1)} \left(\sum_{k \in \mathbb{N}} \big|\tilde{f}(q_{k+1}) - \tilde{f}(q_k)\big| + 4 \sum_{k \in \mathbb{N}} \big(f^+(q_k) - f^-(q_k)\big) \right)$$
$$\leq 2^{-(i+1)} \big((M + \frac{8}{3}) + 4 \cdot \frac{8}{3}\big) = 2^{-(i+1)} (M + \frac{40}{3}).$$

Here, the third inequality is true because we have $f^-(q_k) < \tilde{f}(q_k) < f^+(q_k)$ for every k (in particular, for $q_k = q^n_{h_i}$ and $q_k = q^n_{i_j}$ for all $j \in \{0, \ldots, s\}$) by right part of (17). The fourth inequality holds because every rational in the enumeration $q_0, q_1, \ldots$ can be listed among $q^n_0, \ldots, q^n_n$ at most twice, and thus also among $q^n_{i_0}, q^n_{h_0}, \ldots, q^n_{i_s}, q^n_{h_s}$ at most four times. The last inequality holds by (19) and (18).

The same upper bound for the Lebesgue measure of the union of intervals in $Cover(T^n_i)$ can be argued likewise.

The proof of Claim 28

We prove that the rational function $\tilde{f}$ fulfills for all rationals $q \in [0, \beta)$ the inequality

$$|\alpha - \tilde{f}(q)| < 2^{\tilde{i}}(\beta - q) \tag{112}$$

by assuming the converse, i.e., that there exists a rational p such that

$$|\alpha - \tilde{f}(p)| \geq 2^{\tilde{i}}(\beta - p). \tag{113}$$

Then, due to the density of $\mathrm{dom}(f)$ on the interval $[0, \beta)$ and the property (20), there exists a rational $q \in (p, \beta)$ such that

$$|\tilde{f}(q) - \tilde{f}(p)| \geq 2^{\tilde{i}-1}(q - \beta). \tag{114}$$

In case $\tilde{f}(q) - \tilde{f}(p) \geq 2^{\tilde{i}-1}(q - \beta)$, we define a finite test

$$Q := (\begin{pmatrix} q \\ f^-(q) \end{pmatrix}, \begin{pmatrix} q \\ f^+(q) \end{pmatrix}, \begin{pmatrix} p \\ f^-(p) \end{pmatrix}, \begin{pmatrix} p \\ f^+(p) \end{pmatrix})$$

and consider the finite test $S := Test_{\tilde{i}-1}(Q)$.

On one hand, it holds that $\beta \in R[\begin{pmatrix} q \\ f^-(q) \end{pmatrix}, \begin{pmatrix} p \\ f^+(p) \end{pmatrix}]$ (and thus also by construction that $\beta \in Cover(S)$) since

$$p < \beta < q + 2^{\tilde{i}-1}\big(\tilde{f}(p) - \tilde{f}(q)\big) < q + 2^{\tilde{i}-1}\big(f^+(p) - f^-(q)\big),$$

where the second inequality holds by (114), . On another hand, both rationals p and q lie on the interval $[0, \beta)$, wherein the enumeration $q_0, q_1, \ldots$ contains inter alia all rationals in that interval, hence there exists two indexes m and k such that $q = q_m$ and $p = q_k$. Then we obtain $Q \subseteq Q_{\max\{m,k\}}$, which implies by Claim 24 that $S \subseteq S^{\max\{m,k\}}_{\tilde{i}-1}$). Therefore, we have

$$\beta \in Cover(S_Q) \subseteq Cover(S^{\max\{m,k\}}_{\tilde{i}-1}) \subseteq Cover(S_{\tilde{i}-1}),$$

which contradicts (34).

The case $\tilde{f}(p) - \tilde{f}(q) \geq 2^{\tilde{i}-1}(q - p)$ leads to contradiction likewise for the test set

$$Q' := (\begin{pmatrix} q \\ 1 - f^+(q) \end{pmatrix}, \begin{pmatrix} q \\ 1 - f^-(q) \end{pmatrix}, \begin{pmatrix} p \\ 1 - f^+(p) \end{pmatrix}, \begin{pmatrix} p \\ 1 - f^-(p) \end{pmatrix})$$

and the finite test $S' := Test_{\tilde{i}-1}(Q')$.

The proof of Claim 29

Now, our goal is to obtain from (112) the similar inequality for the real function f:

$$|\alpha - f(x)| < 2^{\tilde{i}+1}(\beta - x). \tag{115}$$

As in the latter proof, we will do it by contradiction: assuming the existence of a real $x \in [0, \beta)$ such that

$$|\alpha - f(\tilde{x})| \geq 2^{\tilde{i}+1}(\beta - \tilde{x}), \tag{116}$$

we fix k such that $\beta - \tilde{x} \in [2^{-(k+1)}, 2^{-k})$.

The density of the sequence $q_0, q_1, \ldots$ on the interval $[0, \beta)$ and the continuity of the function f on it allow to fix an index $j \geq k + 5$ such that

$$q_j \in (\tilde{x} - \frac{\beta - \tilde{x}}{4}, \tilde{x} - \frac{\beta - \tilde{x}}{4}) \quad \text{and} \tag{117}$$

$$f(q_j) \in \big(f(\tilde{x}) - \frac{\alpha - f(\tilde{x})}{4}, \frac{\alpha - f(\tilde{x})}{4}\big). \tag{118}$$

Then, on one hand, it holds that

$$|f(q_j) - \tilde{f}(q_j)| \leq 2^{-j} \leq \frac{1}{2^5}(\beta - \tilde{x}) \leq \frac{1}{2^5} \cdot 2^{\tilde{i}} \cdot \frac{4}{3}(\beta - q_j) < \frac{1}{16}2^{\tilde{i}}(\beta - q_j). \tag{119}$$

Here, from left to right, the first inequality holds by (17), the second one by choice of k, and the third one by $2^{\tilde{i}} \geq 1$ and (117).

On the other hand, we have

$$|\alpha - f(q_j)| \geq \frac{3}{4}|\alpha - f(\tilde{x})| \geq \frac{3}{4}2^{\tilde{i}+1} \geq \frac{3}{4}2^{\tilde{i}+1} \cdot \frac{3}{4}(\beta - q_j) = \frac{9}{16}2^{\tilde{i}+1}(\beta - q_j). \tag{120}$$

Here, the first inequality holds by (118), the second one by (116), and the third one by (117).

From (119) and (120) together, we obtain that

$$|\alpha - \tilde{f}(q_j)| \geq |\alpha - f(q_j)| - |f(q_j) - \tilde{f}(q_j)| \geq \frac{9}{16}2^{\tilde{i}+1}(\beta - q_j) - \frac{1}{16}2^{\tilde{i}+1}(\beta - q_j) = 2^{\tilde{i}}(\beta - q_j),$$

which contradicts (112).

D The Proof of Claim 30

In this section, we show that, for x converging to β from below, the fraction $\frac{\alpha - f(x)}{\beta - x}$ converges, i.e., that

$$\exists \lim_{q \nearrow \beta} \frac{\alpha - f(q)}{\beta - q}, \tag{121}$$

by contradiction. By Paragraph 5.2, for all $x \in [0, \beta)$, the fraction $\frac{|\alpha - f(x)|}{\beta - f(x)}$ is bounded, consequently, supposing that the left limit in (121) does not exist, we can fix two rational constants c and d where

$$c < d, \quad d - c < 1, \quad \text{and} \quad \liminf_{x \nearrow \beta} \frac{\alpha - f(x)}{\beta - x} < c < d < \limsup_{x \nearrow \beta} \frac{\alpha - f(x)}{\beta - x} \tag{122}$$

and the rational

$$e = d - c > 0. \tag{123}$$

In particular, we obtain from (122) and the right part of (17), we obtain that

$$\liminf_{q\nearrow\beta} \frac{\alpha - f^{+}(q)}{\beta - q} < c < d < \limsup_{q\nearrow\beta} \frac{\alpha - f^{-}(q)}{\beta - q}. \tag{124}$$

Remind that, according to the discussion in Paragraph 5.1, the rational functions f^- and f^+ are totally computable on $LC(\beta)$.

Multiple covering

For a finite test A, its COVERING FUNCTION is

$$k_A \colon [0,1] \longrightarrow \mathbb{N},$$
$$x \longmapsto \#\{i \in \{0,\dots,m\} : x \in U_i\},$$

that is, $k_A(x)$ is the number of intervals in A that contain the real number x. Furthermore, the MEASURE of A is $\mu(A) = \sum_{i\in\{0,\dots,m\}} \mu(U_i)$.

It is easy to see that the measure of a given finite test A can be computed by integrating its covering function on the whole domain $[0,1]$, i.e., for every finite test A, it holds that

$$\mu(A) = \int_0^1 k_A(x)dx, \tag{125}$$

as follows by induction on the number of intervals contained in the finite test A.

The induction base holds true because, in case $A = \emptyset$, we obviously have

$$\mu(A) = 0 = \int_0^1 0dx = \int_0^1 k_A(x)dx$$

and, in case $A = (U)$ is a singleton, the function k_A is just the indicator function of U, while the induction step follows from additivity of the integral operator because the function $k_{(U_0,\dots,U_{n+1})}$ is the sum of $k_{(U_0,\dots,U_n)}$ and $k_{(U_{n+1})}$.

Observe that by our definition of covering function, the values of the covering functions of the two tests $([0.2,0.3],[0.3,0.7])$ and $([0.2,0.7])$ differ on the argument 0.3. Furthermore, for a given finite test and a rational q, by adding intervals of the form $[q,q]$ the value of the corresponding covering function at q can be made arbitrarily large without changing the measure of the test. However, these observations will not be relevant in what follows since they relate only to the value of covering functions at rationals.

D.1 Outline of the proof

For a given finite subset Q of the domain of $\tilde{f}$ (which coincides with the domains of f^- and f^+), we will construct a finite test $M_2(Q)$ by an extension of a construction used by Miller [8, Lemma 1.2] in the left-c.e. case. The construction is effective in the sense that it always terminates and yields the test $M_2(Q)$ in case it is applied to a finite subset of the domain of $\tilde{f}$.

Further, for every finite subset Q of the domain of $\tilde{f}$ and every rational p, we let

$$\tilde{k}_Q(p) = k_{M_2(Q)}(p) \qquad \text{and} \qquad K_Q(p) = \max_{H \subseteq Q} \tilde{k}_H(p). \tag{126}$$

The desired contradiction can be obtained from the following three claims.

▷ Claim 38. Let $Q_0 \subseteq Q_1 \subseteq \cdots$ be a sequence of finite sets that converges to the domain of $\tilde{f}$. Then it holds that

$$\lim_{n\to\infty} K_{Q_n}(e\beta) = \infty.$$

▷ Claim 39. There exists a constant V such that, for every finite subset Q of the domain of $\tilde{f}$, it holds that

$$\int_0^1 \tilde{k}_Q(x)dx = \mu\big(M_2(Q)\big) \leq V. \tag{127}$$

▷ Claim 40. For every finite subset Q of the domain of $\tilde{f}$ and for every nonrational real p in $[0, e]$, it holds that

$$K_Q(p) \leq \tilde{k}_Q(p) + 1. \tag{128}$$

We fix an effective enumeration without repetition $p_0, p_1, \ldots$ of the domain of $\tilde{f}$ and set $Q_n = \{p_0, \ldots, p_n\}$ for $n = 0, 1, \ldots$. We consider a special type of step function with domain $[0, 1]$ that is given by a partition of the unit interval into finitely many intervals with rational endpoints such that the function is constant on the corresponding open intervals but may have arbitrary values at the endpoints. For the scope of this proof, a designated interval of such a step function is an interval that is the closure of a maximum contiguous open interval on which the function attains the same value. I.e., the designated intervals form a partition of the unit interval except that two designated intervals may share an endpoint. Observe that, for every finite subset H of the domain of $\tilde{f}$, the corresponding cover function $\tilde{k}_H(\cdot)$ is such a step function with values in the natural numbers, and the same holds for the function K_{Q_n} since Q_n has only finitely many subsets. Furthermore, for given n, the designated intervals of the function $K_{Q_n}(\cdot)$ together with the endpoints and function value of every interval are given uniformly effective in n because $\tilde{f}$ is computable, and the construction of $M(Q_n)$ is uniformly effective in n.

For all natural numbers i and n, consider the step function K_{Q_n} and its designated intervals. For every such interval, call its intersection with $[0, e]$ its restricted interval. Let X_i^n be the union of all restricted designated intervals where on the corresponding designated interval the function K_{Q_n} attains a value that is strictly larger than 2^{i+2}. Let X_i be the union of the sets $X_i^0, X_i^1, \ldots$.

By (127), applied for Q_n for all natural n, the integral of $\tilde{k}_{Q_n}(p)$ from 0 to 1 is at most V, hence by (128), the integral of $K_{Q_n}(p)$ from 0 to e is at most $V + 1$. Consequently, for a fixed w such that $2^w > V + 1$, each set X_i^n has Lebesgue measure of at most $2^{-(i+w)}$. The latter upper bound then also holds for the Lebesgue measure of the set X_i for every i since, by the maximization in the definition of K_{Q_n} and

$$Q_0 \subseteq Q_1 \subseteq \cdots, \quad \text{we have} \quad K_{Q_0} < K_{Q_1} < \cdots, \qquad \text{hence} \qquad X_i^0 \subseteq X_i^1 \subseteq \cdots.$$

By construction, for all i and $n > 0$, the difference $X_i^n \setminus X_i^{n-1}$ is equal to the union of finitely many intervals that are mutually disjoint except possibly for their endpoints, and a list of these intervals is uniformly computable in i and n since the functions K_{Q_n} are uniformly computable in n. Accordingly, the set X_i is equal to the union of a set U_i of intervals with rational endpoints that is effectively enumerable in i and where the sum of the measures of these intervals is at most $2^{-(i+w)}$. By the two latter properties, the sequence $U_0, U_1, \ldots$ is a Martin-Löf test. By Claim 38, the values $K_{Q_n}(e\beta)$ tend to infinity where $e\beta < e$, hence for all n, the Martin-Löf random real $e\beta$ is contained in some interval in U_n, a contradiction. This concludes the proof that Claim 38 through 40 together imply that the left limit (121) exists.

It remains to construct the finite Test $M_2(Q)$ for a given finite subset Q of the domain of g and check that Claims 38 through 40 are fulfilled.

Preliminaries fo the test construction: the functions $\gamma(q)$ and $\delta(q)$

First, we define two partial computable functions γ and δ that have the same domain as $\tilde{f}$:

$$\gamma(q) = \tilde{f}(q) - cq \qquad \text{and} \qquad \delta(q) = \tilde{f}(q) - dq.$$

Due to $c < d$, the following claim is immediate:

▷ Claim 41. For every q in the domain of $\tilde{f}$, we have

$$\gamma(q) - \delta(q) > (d-c)q = eq > 0, \qquad \text{hence} \qquad \gamma(q) > \delta(q).$$

In particular, the partial function $\gamma - \delta$ is strictly increasing on its domain, hence, for every sequence $q_0 < q_1 < \ldots$ of rationals in $[0, \beta)$ that converges to β, the values $g(q_i)$ are defined, and therefore, the values $\gamma(q_i) - \delta(q_i)$ converge strictly increasingly to $(d-c)\beta$.

Outline of the construction of the finite test $M_2(Q)$

Let $Q = \{q_0 < \cdots < q_n\}$ be a nonempty finite subset of the domain of $\tilde{g}$, i.e., $q_0 = p_{k_0}$, $q_1 = p_{k_1}$, ..., $q_n = p_{k_n}$ for appropriate indexes $k_0, \ldots, k_n$, where the notation used to describe Q has its obvious meaning, i.e., Q is the set of $q_0, \ldots, q_n$, and $q_i < q_{i+1}$ for all i. Note that — in contrast to Paragraph 5.2 — $q_0, \ldots, q_n$ don't need to be *first* $n+1$ enumeration elements $p_0, \ldots, p_n$ rearranged. We describe the construction of the finite test $M_2(Q)$, which is an extended version of a construction used by Titov [13, Theorem 66] in connection with a nondecreasing $\mathbb{Q}$-translation function. Using the notation defined in the previous paragraphs, for all i in $\{0, \ldots, 2n+1\}$, let

$$\begin{aligned}
\delta_{2i} &= \delta(q_i) - 2^{-2k_i+1} = g(q_i) - dq_i + 2^{-2k_i+1},\\
\gamma_{2i} &= \gamma(q_i) - 2^{-2k_i+1} = g(q_i) - cq_i + 2^{-2k_i+1},\\
\delta_{2i+1} &= \delta(q_i) + 2^{-2k_i+1} = g(q_i) - dq_i - 2^{-2k_i+1},\\
\gamma_{2i+1} &= \gamma(q_i) + 2^{-2k_i+1} = g(q_i) - cq_i - 2^{-2k_i+1},
\end{aligned}$$

and, for all i, j in the range $0, \ldots, 2i+1$, define

$$J[i,j] = [\gamma_i - \delta_i, \gamma_j - \delta_i] = [eq_i, \gamma_j - \delta_i]. \tag{129}$$

Then we set

$$M_2(Q) = M((\delta_0, \gamma_0), (\delta_1, \gamma_1), \ldots, (\delta_{2n+1}, \gamma_{2n+1})), \tag{130}$$

where $M((\delta_0, \gamma_0), (\delta_1, \gamma_1), \ldots, (\delta_{2n+1}, \gamma_{2n+1}))$ is the result of extended Miller algorithm formally described in [13, pp. 58-59] applied on the set $M((\delta_0, \gamma_0), (\delta_1, \gamma_1), \ldots, (\delta_{2n+1}, \gamma_{2n+1}))$.

We define it explicitly in next paragraph.

Outline of the construction of the extended Miller algorithm

For a fixed m and fixed set of pairs $M((\delta_0,\gamma_0),\dots,(\delta_m,\gamma_m))$ fulfilling

$$\gamma_0-\delta_0\le\dots\le\gamma_m-\delta_m \text{ and} \tag{131}$$
$$\text{all } \tfrac{\gamma_i-\delta_i}{d-c} \text{ lie in the domain of } \tilde{f}, \tag{132}$$

the test $M((\delta_0,\gamma_0),\dots,(\delta_m,\gamma_m))$ is constructed in successive steps $j=0,1,\dots,m$, where, at each step j, intervals $U_0^j,\dots,U_m^j$ are defined. Every such interval U_i^j has the form

$$U_i^j=J[i,\boldsymbol{r}^j(i)]=J[i,k]=[\gamma_i-\delta_i,\gamma_k-\delta_i]$$

for some index $k\in\{0,\dots,m\}$, where $\boldsymbol{r}^j(\cdot)$ is an index-valued function that maps every index i to such index k that $J[i,k]=U_i^j$.

At step 0, for $i=0,\dots,m$, we set the values of the function $\boldsymbol{r}^0(i)$ by

$$\boldsymbol{r}^0(i)=i \tag{133}$$

and initialize the intervals U_i^0 as zero-length intervals

$$U_i^0=J[i,\boldsymbol{r}^0(i)]=J[i,i]. \tag{134}$$

In the subsequent steps, every change of an interval amounts to an expansion at the right end in the sense that, for all indices i, the intervals $U_i^0,\dots,U_i^m$ share the same left endpoint, while their right endpoints are nondecreasing. More precisely, as we will see later, for $i=0,\dots,m$, we have

$$\gamma_i-\delta_i=\min U_i^0=\dots=\min U_i^m,$$
$$\max U_i^0\le\dots\le\max U_i^m,$$

and thus $U_i^0\subseteq\dots\subseteq U_i^m$. After concluding step m, we define the finite test

$$M(P)=(U_0^m,\dots,U_m^m).$$

In case the right endpoints of two intervals of the form U_i^{j-1} and U_i^j coincide, we say that the interval with index i remains unchanged at step j. Similarly, we will speak informally of the interval with index i, or U_i, for short, in order to refer to the sequence $U_i^0,\dots,U_i^m$ in the sense of one interval that is successively expanded.

Due to technical reasons, for an empty set $\emptyset$, we define $M_2(\emptyset)=\emptyset$.

A single step of the construction and the index stair

During step $j>0$, we proceed as follows. Let t_0 be the largest index among $\{0,\dots,j-1\}$ such that $\gamma_{t_0}>\gamma_j$, i.e., let

$$t_0=\arg\max\{q_z : z<j \text{ and } \gamma_z>\gamma_j\} \tag{135}$$

in case such index exists and $t_0=-1$ otherwise.

Next, define indices $s_1,t_1,s_2,t_2,\dots$ inductively as follows. For $h=1,2,\dots$, assuming that t_{h-1} is already defined, where $t_{h-1}<j-1$, let

$$s_h = \max\ \arg\min\{\delta_x : t_{h-1}<x\le j-1\}, \tag{136}$$
$$t_h = \max\ \arg\max\{\gamma_y : s_h\le y\le j-1\}. \tag{137}$$

That is, the operator arg min yields a set of indices x such that δ_x is minimum among all considered values, and s_h is chosen as the largest index in this set, and similarly for arg max and the choice of t_h.

Since we assume that $t_{h-1} < j-1$, the minimization in (136) is over a nonempty set of indices, hence s_h is defined and satisfies $s_h \leq j-1$ by definition. Therefore, the maximization in (137) is over a nonempty index set, hence also t_h is defined.

The inductive definition terminates as soon as we encounter an index $l \geq 0$ such that $t_l = j-1$, which will eventually be the case by the previous discussion and because, obviously, the values $t_0, t_1, \dots$ are strictly increasing. For this index l, we refer to the finite sequence $(t_0, s_1, t_1, \dots, s_l, t_l)$ (or, for short, $(t_0, s_1, t_1, \dots)$ in case the value of l is not important) as the INDEX STAIR OF STEP j. E.g., in case $l = 1$, the index stair is (t_0, s_1, t_1), and in case $l = 0$, the index stair is (t_0). Note that $l = 0$ holds if and only if even s_1 could not be defined, where the latter in turn holds if and only if t_0 is equal to $j-1$.

Next, for $i = 1, \dots, m$, we set the values of $\boldsymbol{r}^j(i)$ and define the intervals U_i^j. For a start, in case $l \geq 1$, let

$$\begin{aligned} \boldsymbol{r}^j(s_1) &= j, && (138)\\ U_{s_1}^j &= J[s_1, \boldsymbol{r}^j(s_1)] = J[s_1, j] = [\gamma_{s_1} - \delta_{s_1}, \gamma_j - \delta_{s_1}], && (139) \end{aligned}$$

and call this a NONTERMINAL EXPANSION of the interval U_{s_1} AT STEP j. In case $l \geq 2$, in addition, let for $h = 2, \dots, l$

$$\begin{aligned} \boldsymbol{r}^j(s_h) &= t_{h-1}, && (140)\\ U_{s_h}^j &= J[s_h, \boldsymbol{r}^j(s_h)] = J[s_h, t_{h-1}] = [\gamma_{s_h} - \delta_{s_h}, \gamma_{t_{h-1}} - \delta_{s_h}], && (141) \end{aligned}$$

and call this a TERMINAL EXPANSION of the interval U_{s_h} AT STEP j.

For all remaining indices, the interval with index i REMAINS UNCHANGED AT STEP j, i.e., for all $i \in \{0, \dots, n\} \setminus \{s_1, \dots, s_l\}$, let

$$\begin{aligned} \boldsymbol{r}^j(i) &= \boldsymbol{r}^{j-1}(i), && (142)\\ U_i^j &= U_i^{j-1}. && (143) \end{aligned}$$

The choice of the term "terminal expansion" is motivated by the fact that, in case a terminal expansion occurs for the interval with index i at step j, then, at all further steps $j+1, \dots, m$, the interval remains unchanged, as we will see later.

We conclude step j by defining for $i = 0, \dots, m$ the half-open interval

$$V_i^j = U_i^j \setminus U_i^{j-1}. \tag{144}$$

That is, during step j, the interval with index i is expanded by adding at its right end the half-open interval V_i^j, i.e., we have

$$U_i^j = U_i^{j-1} \dot\cup V_i^j \qquad \text{where} \qquad |U_i^j| = |U_i^{j-1}| + |V_i^j|. \tag{145}$$

This includes the degenerated case where the interval with index i is not changed, hence V_i^j is empty and has length 0.

In what follows, in connection with the construction of a test of the form $M_2(Q)$, when appropriate, we will occasionally write t_0^j for the value of t_0 chosen during step j and similarly for other values like s_h in order to distinguish the values chosen during different steps of the construction.

The proof of Claim 38 through 40

Claim 38 follows in the same way as [13, Theorem 66, Claim 1] for the sets

$$S = \{q < \beta\colon \frac{\alpha - f^-(q)}{\beta - q} > d\},$$

$$T = \{q < \beta\colon \frac{\alpha - f^+(q)}{\beta - q} < c\}.$$

In order to prove Claim 40, in a similar manner to (126), we define

$$\tilde{k}'_P(p) = k_{M(P)}(p) \qquad \text{and} \qquad K'_P(p) = \max_{H \subseteq P} \tilde{k}'_H(p). \tag{146}$$

for every set of pairs P that fulfills (131) and (132).

Then Claim 40 is easily implied by the corresponding property of extended Miller algorithm that we remind in what follows.

▷ Claim 42. For every set of pairs of rationals P that fulfills (131)-(132) and, for every nonrational real p in $[0, e]$, it holds that

$$K'_P(p) \leq \tilde{k}'_P(p) + 1. \tag{147}$$

Proof. See [13, Theorem 66, Claim 3]. ◀

Then we prove Claim 40 by contradiction: supposing the existence of a point p, a set $Q = (q_0, \dots, q_n)$ and its subset $H = (q_{h_0}, \dots, q_{h_m})$ such that

$$\tilde{k}_H(p) = K_Q(p) \geq \tilde{k}_Q(p) + 2,$$

then the sets of pairs

- $P = (\delta_0, \gamma_0), (\delta_1, \gamma_1), \dots, (\delta_{2n+1}, \gamma_{2n+1})$, where $\delta_{2i} = \delta(q_i) + 2^{-2k_i+1}, \gamma_{2i} = \gamma(q_i) + 2^{-2k_i+1}$, $\delta_{2i+1} = \delta(q_i) - 2^{-2k_i+1}$, and $\gamma_{2i+1} = \gamma(q_i) - 2^{-2k_i+1}$ for every $i \in \{0, \dots, n\}$,

and its subset

- $P_H = (\delta'_0, \gamma'_0), (\delta'_1, \gamma'_1), \dots, (\delta'_{2m+1}, \gamma'_{2m+1})$ where $\delta'_{2i} = \delta(q_{h_i}) + 2^{-2k_{h_i}+1}, \gamma_{2i} = \gamma(q_{h_i}) + 2^{-2k_{h_i}+1}$, $\delta_{2i+1} = \delta(q_{h_i}) - 2^{-2k_{h_i}+1}$ for every $i \in \{0, \dots, m\}$,

both fulfill (131) and (132), hence by definition (130), we obtain that

$$K'_P(p) \geq \tilde{k}'_{P_H}(p) = \tilde{k}_H(p) \geq \tilde{k}_Q(p) + 2 = \tilde{k}'_P(p) + 2,$$

which contradicts to Claim 42.

Before proving Claim 39, we remind some structural properties of the extended Miller algorithm.

▷ Claim 43. Let $j > 0$ be a step of the construction of $M_2(Q)$..

If $\gamma_{j-1} \leq \gamma_j$, then it holds for the index stair $(t_0, s_1, t_1, \dots, s_l, t_l)$ of this step that $l > 0$, i.e., that s_1 can be defined, and we have

$$V^j_{s_1} = (\gamma_{t_1} - \delta_{s_1}, \gamma_j - \delta_{s_1}], \tag{148}$$

$$V^j_{s_h} = (\gamma_{t_h} - \delta_{s_h}, \gamma_{t_{h-1}} - \delta_{s_h}] \quad \text{for } h \geq 2 \quad \text{(if defined)}, \tag{149}$$

$$V^j_i = \emptyset \quad \text{for } i \text{ in } \{0, \dots, n\} \setminus \{s_1, \dots, s_l\}. \tag{150}$$

In particular, the half-open intervals $V_0^j, \dots, V_n^j$ are mutually disjoint, and the sum of their Lebesgue measures can be bounded as follows

$$\sum_{i=0}^{n} \mu(V_i^j) = \sum_{h=1}^{l} \mu(V_{s_h}^j) = \gamma_j - \gamma_{j-1}. \tag{151}$$

If $\gamma_{j-1} > \gamma_j$, then the index stair of this step has a form $(j-1)$, i.e., $t_0 = j-1$, $l = 0$, all the intervals $V_0^j, \dots, V_n^j$ are empty, i.e.,

$$V_i^j = \emptyset \text{ for all } i, \tag{152}$$

and the sum of their Lebesgue measures is equal to zero

$$\sum_{i=0}^{n} \mu(V_i^j) = 0. \tag{153}$$

Proof. See [13, Theorem 66, Claim 12]. ◀

The proof of Claim 39

Using the results on the intervals V_i^j in Claim 43, we can now easily demonstrate Claim 39. We have to find a constant V such that, for every subset $Q = \{q_0 < \dots < q_n\}$ of the domain of g that

$$\mu\big(M_2(Q)\big) \le V. \tag{154}$$

Set $V = V_{\tilde f} + 4$ where $V_{\tilde f}$ is (by Lemma 19, finite) total variation of $\tilde f$.

For the set $P = (\delta_0, \gamma_0), (\delta_1, \gamma_1), \dots, (\delta_{2n+1}, \gamma_{2n+1})$, where $\delta_{2i} = \delta(q_i) + 2^{-2k_i+1}, \gamma_{2i} = \gamma(q_i) + 2^{-2k_i+1}$, $\delta_{2i+1} = \delta(q_i) - 2^{-2k_i+1}$, and $\gamma_{2i+1} = \gamma(q_i) - 2^{-2k_i+1}$ for every i, we have $M_2(Q) = M(P)$, hence

$$\begin{aligned} \mu\big(M_2(Q)\big) &= \sum_{U \in M(P)} \mu(U) = \sum_{i=0}^{2n+1} \mu(U_i^n) = \sum_{i=0}^{2n+1} \sum_{j=1}^{2n+1} \mu(V_i^j) = \sum_{j=1}^{2n+1} \sum_{i=0}^{2n+1} \mu(V_i^j) \\ &= \sum_{j=1}^{2n+1} \big(\max\{\gamma_j - \gamma_{j-1}, 0\}\big) \le \sum_{i=1}^{n} |\tilde f(q_i) - f(q_{i-1})| + 2 \sum_{k=0}^{\infty} 2^{-(2k-1)} < V_f + 4 \end{aligned}$$

In the first line, the first equality holds by definition of $\mu\big(M_2(Q)\big)$, while the second and the third equalities hold by construction of $M_2(Q)$ and by (144), respectively.

In the second line, the equality holds because, for every j, we have

$$\sum_{i=0}^{n} \mu(V_i^j) = \max\{\gamma_j - \gamma_{j-1}, 0\}$$

due to the following argument: in case $\gamma_{j-1} \le \gamma_j$, we obtain from Claim 43, (151), that $\sum_{i=0}^{n} \mu(V_i^j) = \gamma_j - \gamma_{j-1} \ge 0$, and in case $\gamma_{j-1} > \gamma_j$, we obtain from Claim 43, (153), that $\sum_{i=0}^{n} \mu(V_i^j) = 0$. The second inequality in the second line holds by definition of variation and Lemma 19.

If remains to prove the first inequality in the second line. For $j = 2i+1$, we fix $q_i = p_{k_i}$ and directly obtain

$$\gamma_j - \gamma_{j-1} = \big(\tilde f(q_i) - cq_i - 2^{-(2k_i-1)}\big) - \big(\tilde f(q_i) - cq_i + 2^{-(2k_i-1)}\big) < 0,$$

hence $\max\{\gamma_j - \gamma_{j-1}, 0\} = 0$.

For $j = 2i+1$, we fix $q_{i_1} = p_{k_{i-1}}$ $q_i = p_{k_i}$, we obtain

$$\begin{aligned}\gamma_j - \gamma_{j-1} &= \big(\tilde{f}(q_i) - cq_i + 2^{-(2k_i-1)}\big) - \big(\tilde{f}(q_{i-1}) - cq_{i-1} + 2^{-(2k_{i-1}-1)}\big)\\ &= \big(f(q_i) - f(q_{i-1})\big) + \underbrace{c(q_{i-1} - q_i)}_{\leq 0} + \big(2^{-(2k_{i-1}-1)} + 2^{-(2k_i-1)}\big),\end{aligned}$$

hence $\max\{\gamma_j - \gamma_{j-1}, 0\} \leq \tilde{f}(q_i) - \tilde{f}(q_{j-1}) + 2^{-(2k_{i-1}-1)} + 2^{-(2k_i-1)}$.