Variants of Solovay reducibility

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Abstract. Solovay reducibility was introduced by Robert M. Solovay [7] in 1975 in terms of translation functions on rationals. It is a measure of relative approximation speed and thus also of relative randomness of reals. In the area of algorithmic randomness, Solovay reducibility has been intensively studied and several central results on left-c.e. reals have been obtained.

Outside of the left-c.e. reals, Solovay reducibility is considered to be behaved badly [2]. Proposals for variants of Solovay reducibility that are better suited for the investigation of arbitrary, not necessarily left-c.e. reals were made by Rettinger and Zheng [11], and, recently, by Titov [8] and by Kumabe and co-authors [4], [5]. These variants all coincide with the original version of Solovay reducibility on the left-c.e. reals. Furthermore, they are all defined in terms of translation functions. The latter translate between computable approximations in the case of Rettinger and Zheng, are monotone in the case of Titov, and are functions between reals in the case of Kumabe et al.

In what follows, we derive new results on the mentioned variants and their relation to each other. In particular, we obtain that Solovay reducibility defined in terms of translation function on rationals implies Solovay reducibility defined in terms of translation functions on reals, and we show that the original version of Solovay reducibility is strictly weaker than its monotone variant.

Solovay reducibility and its variants mentioned so far have tight connections to Martin-Löf randomness, the strongest and most central notion of a random sequence. For the investigation of Schnorr randomness, total variants of Solovay reducibility have been introduced by Merkle and Titov [6] in 2022 and, independently, by Kumabe et al. [5] in 2024, the latter again via real-valued translation functions. In what follows, we show that total Solovay reducibility defined in terms of rational functions implies total Solovay reducibility defined in terms of real functions and is strictly stronger than the original version of Solovay reducibility.

1 Solovay reducibility and its variants

We start with reviewing the concept of Solovay reducibility introduced by Solovay [7] in 1975 as a measure of relative randomness. Its original definition uses the notion of a translation function on rationals, or a $\mathbb{Q}$-translation function, defined on the left cut of a real. Our notation is standard. All rationals and reals are supposed to be on the interval $[0, 1)$ if not stated otherwise. A left-c.e.

approximation is a strictly increasing computable approximation. Unexplained notation can be found in Downey and Hirschfeldt [2].

1.1 Solovay reducibility

Definition 1 (Solovay, 1975). *The* LEFT CUT *of a real β is the set* $\mathrm{LC}(\beta)$ *of all rationals y where $0 \leq y < \beta$. A* $\mathbb{Q}$-TRANSLATION FUNCTION *from a real β to a real α is a partially computable function g from the set $\mathbb{Q} \cap [0,1)$ to itself such that, for every $q \in \mathrm{LC}(\beta)$, the value $g(q)$ is defined and fulfills $g(q) < \alpha$, and it holds that*

$$\lim_{q \nearrow \beta} g(q) = \alpha, \tag{1}$$

where $\lim\limits_{q \nearrow \beta}$ denotes the left limit.

A real α is SOLOVAY REDUCIBLE *to a real β, also written as $\alpha \leq_{\mathrm{S}} \beta$, if there is a real constant c and a $\mathbb{Q}$-translation function g from β to α such that for all $q < \beta$, it holds that*

$$\alpha - g(q) < c(\beta - q). \tag{2}$$

A $\mathbb{Q}$-translation function maps rationals to rationals — in contrast to $\mathbb{R}$-translation functions, to be introduced in the next section, which map reals to reals. We refer to inequality (2) as SOLOVAY CONDITION and to the constant c that occurs in it as SOLOVAY CONSTANT.

Recall that a function f is LIPSCHITZ CONTINUOUS on some interval I if there exists a constant d, called LIPSCHITZ CONSTANT, such that for every $p, q \in I$ in the domain of f, it holds that $|f(q) - f(p)| < d|q - p|$.

The Solovay condition (2) resembles a localized version of the defining condition of Lipschitz continuity, where instead of arbitrary pairs of arguments, we consider only pairs with second component β. Accordingly, there are close relations between $\mathbb{Q}$-translation functions that are Lipschitz continuous and ones that witness Solovay reducibility.

Proposition 1. *Let α and β be two reals.*

1. *If g is a Lipschitz continuous $\mathbb{Q}$-translation function from β to α, then $\alpha \leq_{\mathrm{S}} \beta$ via g.*
2. *If α and β are left-c.e. and $\alpha \leq_{\mathrm{S}} \beta$, then $\alpha \leq_{\mathrm{S}} \beta$ via some Lipschitz continuous $\mathbb{Q}$-translation function. Moreover, the function can be chosen strictly increasing.*

Proof. 1. The proof is standard and is left to the interested reader.

2. By [2, Proposition 9.1.7], there exist a constant $d > c$ and two left-c.e. approximations $a_0, a_1, \dots$ and $b_0, b_1, \dots$ of α and β, respectively, that fulfill

$$a_{n+1} - a_n < d(b_{n+1} - b_n) \quad \text{for every } n. \tag{3}$$

Then, the function g from $\mathrm{LC}(\beta)$ to $\mathrm{LC}(\alpha)$ defined by

$$g(q) = a_n + \frac{a_{n+1} - a_n}{b_{n+1} - b_n}(q - b_n) \quad \text{for all rationals } q \in [b_n, b_{n+1})$$

and $h(q) = a_0$ for all rationals $q \in [0, a_0)$ is Lipschitz continuous with the Lipschitz constant d by (3). Further, f is strictly increasing since the sequences $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ are strictly increasing. Finally, f fulfills (1) because, for arbitrarily small distance $\beta - b_n$, we have $0 < \alpha - f(q) < d(\beta - b_n)$ for all $q \in [a_n, \alpha)$.
Therefore, f is a Lipschitz continuous strictly increasing $\mathbb{Q}$-translation function from β to α, hence, by the statement (1) of the current proposition, witnesses $\alpha \leq_{\mathrm{S}} \beta$.

□

In 2022, Merkle and Titov introduced [6] a version of Solovay reducibility via a totally defined $\mathbb{Q}$-translation function.

Definition 2 (Merkle, Titov, 2022). *A real α is* TOTAL SOLOVAY REDUCIBLE *to a real β, written $\alpha \leq_{\mathrm{S}}^{\mathrm{tot}} \beta$, if $\alpha \leq_{\mathrm{S}} \beta$ via a $\mathbb{Q}$-translation function g which is totally computable on $[0, 1)$.*

1.2 Monotone Solovay reducibility

In 2024, Titov [8] proposed the following monotone variant of Solovay reducibility, which coincides with $\leq_{\mathrm{S}}$ on the set on left-c.e. reals.

Definition 3. *A real α is* MONOTONE SOLOVAY REDUCIBLE *to a real β, written $\alpha \leq_{\mathrm{S}}^{\mathrm{m}} \beta$, if α is* SOLOVAY REDUCIBLE *to β via a $\mathbb{Q}$-translation function that is monotone nondecreasing.*

By definition, Solovay reducibillity is implied by its monotone variant. In the remainder of this section, we demonstrate that the reverse implication does not hold in general, i.e., $\leq_{\mathrm{S}}^{\mathrm{m}}$ is strictly stronger than $\leq_{\mathrm{S}}$. For a start, we state a somewhat technical observation about the behavior of monotone $\mathbb{Q}$-translation functions, its proof is left to the interested reader.

Proposition 2. *Let g be a $\mathbb{Q}$-translation function from a real β to a real α, and let $q_0, q_1, \ldots$ be a sequence of rationals in $[0, \beta)$ such that $\lim\limits_{n\to\infty} g(q_n) = \beta$. Then $\lim\limits_{n\to\infty} q_n = \alpha$.*

The next proposition shows that the monotone $\mathbb{Q}$-translation function can be only from a left-c.e. to a left-c.e. real or from a nonleft-c.e. to a nonleft-c.e. real.

Proposition 3. *Let $\alpha, \beta \in \mathbb{R}$, where α is left-c.e. Then the following statements are equivalent:*

1. *β is left-c.e.;*
2. *there exists a nondecreasing $\mathbb{Q}$-translation function from α to β;*
3. *there exists a nondecreasing $\mathbb{Q}$-translation function from β to α.*

Proof. Let $a_0, a_1, \ldots$ where $a_0 = 0$ be a left-c.e. approximation of α.

- In case β is left-c.e., let $b_0, b_1, \dots$ where $b_0 = 0$ be a left-c.e. approximation of β. Then the functions

$$g(q) = a_{\min\{m: b_m \geq q\}} \quad \text{and} \quad h(q) = b_{\min\{m: a_m \geq q\}}$$

  are nondecreasing translation functions from β to α and from α to β, respectively.
- If there exists a nondecreasing $\mathbb{Q}$-translation function g from β to α, then the sequence $g(a_0), g(a_1), \dots$ is a nondecreasing computable approximation of β, hence β is left-c.e.
- If there exists a nondecreasing $\mathbb{Q}$-translation function g from β to α, then for a fixed computable enumeration $q_0 = 0, q_1, \dots$ of $\mathrm{dom}(g)$, we define $i_0, i_1, \dots$ step-wise: at step 0, we set $i_0 = 0$; at step $n+1$, we search for an index i such that

$$q_{i_n} < q_i,\ g(q_i) > a_n \text{ and there exists an index } j > i_n \text{ such that } g(q_i) < a_j \tag{4}$$

  and set $i_{n+1} = i$.
  Then every step terminates since, for every n, all rationals in the interval $(q_{i_n}, \beta) \cap (d_n, \beta)$, where d_n is some point in $(0, \beta)$ such that $f(d_n) \in (a_{i_n}, \alpha)$, fulfill (4). Therefore, the sequence $q_{i_0}, q_{i_1}, \dots$ is infinite and computable. It is increasing by $q_{i_n} < q_{i_{i+1}]}$. Finally, it fulfills $g(q_{i_n}) > a_n$ for every n, hence the sequence $g(q_{i_0}), g(q_{i_1}), \dots$ tends to β; and thus, by Proposition 2, the sequence $q_{i_0}, q_{i_1}, \dots$ tends to β. Therefore, the real β is left-c.e.

□

Corollary 1. *The set of left-c.e. reals is closed downwards and upwards in* $\mathbb{R}$ *relative to* $\leq_{\mathrm{S}}^{\mathrm{m}}$.

On the other hand, relative to $\leq_{\mathrm{S}}$, the left-c.e. reals are not closed downwards, hence $\leq_{\mathrm{S}}^{\mathrm{m}}$ is not equivalent to $\leq_{\mathrm{S}}$. We give an example of such two real in the next proposition.

Proposition 4. *There exist a nonleft-c.e. real* β *such that* $1 \leq_{\mathrm{S}} \beta$.

Proof. We fix an enumeration $q_0, q_1, \dots$ of all rationals in the unit interval and define a computable test $I_n := [q_n, q_n + 2^{-2n}]$. Since $\sum_{n=0}^{\infty} \mu(I_n) = \frac{1}{3}$, while the set of all left-c.e. reals has the Lebesgue measure 0 because there are only countably many such reals, we can fix a nonleft-c.e. real $\beta \notin \bigcup_{i \in \mathbb{N}} I_n$.

On the one hand, the function g defined by

$$f(q_n) = 1 - 2^{-2n}$$

is a $\mathbb{Q}$-translation function from β to 1. Moreover, for every $q_n < \beta$, we easily obtain from $\beta \notin I_n$ that $\beta > q_n + \mu(I_n) = q_n + 2^{-2n}$, and thus

$$1 - g(q_n) = 1 - (1 - 2^{-2n}) = 2^{-2n} \in (0, \beta - q_n).$$

Therefore we obtain that $1 \leq_{\mathrm{S}} \beta$ with the constant 1 via g.

On the other hand, the existence of a $\mathbb{Q}$-translation function from 1 to β would imply by Proposition 3 that β is left-c.e., a contradiction. □

Corollary 2. *There exist two reals* α *and* β *such that* $\alpha \leq_{\mathrm{S}} \beta$ *but* $\alpha \not\leq_{\mathrm{S}}^{\mathrm{m}} \beta$.

1.3 Computable functions on reals

Definition 4. *A sequence $q_0, q_1, \ldots$ of rationals is called* EFFECTIVE APPROXIMATION *if it fulfills $|q_n - q_{n+1}| < 2^{-n}$ for every n.*

Since every effective approximation $q_0, q_1, \ldots$ is a Cauchy sequence, it converges to some limit point $x \in \mathbb{R}$. The following properties of effective approximations and their limits can be obtained straightforwardly.

Lemma 1. *1. If $q_0, q_1, \ldots$ is an effective approximation, then there exists a real x such that*

$$|x - q_n| < 2^{-(n-1)} \text{ for all } n \in \mathbb{N}. \tag{5}$$

In particular, $x = \lim_{n\to\infty} q_n$. In that case, we also say that $q_0, q_1, \ldots$ is an EFFECTIVE APPROXIMATION OF x.

2. Let $q_0, q_1, \ldots$ be an approximation of a real x and it holds that

$$|q_{n+1} - q_n| < 2^{-n} \text{ for all } n \in \mathbb{N}. \tag{6}$$

Then $q_0, q_1, \ldots$ is an effective approximation of x.

Obviously, a real x is computable iff there exists a computable effective approximation of x.

The class of Turing machines that, using an infinite sequence of finite strings (in our case, encoded rationals) as an oracle, returns another sequence of finite strings was formally defined by Weihrauch [10, Chapter 2] under the name "Turing machines of Type 2". In what follows, we give a formal definition of a computable real function using a notion of a Turing machine of Type 2 specified for the sequences of rationals.

Definition 5 (Weihrauch, 2000). *A* TURING MACHINE M OF TYPE 2 *is an oracle Turing machine that, for every oracle $(p_0, p_1, \ldots)$, where $p_0, p_1, \ldots$ are (appropriately finitely encoded) rationals, produces either an infinite sequence of rationals $(q_0, q_1, \ldots)$; in this case, we write $M^{(p_0,p_1,\ldots)} \downarrow= (q_0, q_1, \ldots)$ and say that M* RETURNS THE SEQUENCE $(q_0, q_1, \ldots)$ FROM THE INPUT $(p_0, p_1, \ldots)$*; or a finite set of rationals $(q_0, q_1, \ldots, q_n)$; in the latter case, we say that $M^{(p_0,p_1,\ldots)}$* IS UNDEFINED.

A real function f from some subset of $\mathbb{R}$ to $\mathbb{R}$ is COMPUTABLE *on some set $X \subseteq \mathrm{dom}(f)$ if there exists an oracle Turing machine M such that, for every $x \in X$ and every effective approximation $p_0, p_1, \ldots$ that converges to x, $M^{(p_0,p_1,\ldots)}$ returns an effective approximation $q_0, q_1, \ldots$ of $f(x)$.*

By [10, Theorem 4.3.1], computability of a real function implies its continuity.

Proposition 5. *It a real function f is computable on some interval (a, b), then f is continuous on (a, b).*

The following proposition is straightforwardly implied by [10, Corollary 6.2.5].

Proposition 6 (Weihrauch, 2000). *If a real function g is computable on the set $[a, b)$, then the maximum function $h(x) = \max\{g(y) : a \leq y \leq x\}$ is computable.*

1.4 Variants of Solovay reducibility defined via translation function on reals

According to Proposition 1(1), on the left-c.e. reals, the Solovay reducibility is equivalent to the Solovay reducibility via a Lipschitz continuous $\mathbb{Q}$-translation function. In 2020, Kumabe, Miyabe, Mizusawa, and Suzuki proposed [4, Definition 9] a new type of reducibility by replacing the Lipschitz continuous translation functions on rationals in the latter characterization by the translation functions on reals, without additionally requiring for the reals α and β to be left-c.e.

In what follows, we give the formal definition of this reducibility denoted by Kumabe et al. as "L2" under the more intuitive name "real Solovay reducibility".

Definition 6 (Kumabe et al., 2020). *An* $\mathbb{R}$-TRANSLATION FUNCTION *from a real* β *to another real* α *is a real function which is computable on the interval* $[0,\beta)$ *and maps it to the interval* $[0,\alpha)$ *such that*

$$\lim_{x\nearrow\beta} f(x) = \alpha. \tag{7}$$

A real α *is* REAL SOLOVAY REDUCIBLE *to a real* β*, written* $\alpha\leq^{\mathbb{R}}_{\mathrm{S}}\beta$*, if there exists a Lipschitz continuous* $\mathbb{R}$*-translation function* f *from* β *to* α*.*

Note that, on the set on left-c.e. reals, an $\mathbb{R}$-translation function between every two reals always exists.

Proposition 7. *If* α *and* β *are left-c.e. reals, then there exists an* $\mathbb{R}$*-translation function from* β *to* α*.*

Proof. Fixing two left-c.e. approximations $a_0, a_1, \dots \nearrow \alpha$ and $b_0, b_1, \dots \nearrow \beta$, we can construct an $\mathbb{R}$-translation function f from β to α by setting

$$f(x) = a_n + \frac{x-b_n}{b_{n+1}-b_n}(a_{n+1}-a_n) \quad \text{if there exists such } n \text{ that } b_n \leq x < b_{n+1}.$$

□

In contrast to Solovay reducibility (see Proposition 2), the additional requirement for the $\mathbb{R}$-translation function in the latter definition to be nondecreasing (Kumabe et al. denoted [4, Definition 9] the resulting reducibility "L1") does not induce a strictly stronger reducibility on $\mathbb{R}$ than $\leq^{\mathbb{R}}_{\mathrm{S}}$, as we will see in the next proposition.

Proposition 8. *If* f *is a* $\mathbb{R}$*-translation function from a real* β *to another real* α*, then*

$$\tilde{f}(x) = \max\{f(y) : y \in [0,x]\}$$

is a monotone nondecreasing $\mathbb{R}$*-translation function from* β *to* α*.*

Moreover, if $\alpha\leq^{\mathbb{R}}_{\mathrm{S}}\beta$ *via* f*, then* $\alpha\leq^{\mathbb{R}}_{\mathrm{S}}\beta$ *via* $\tilde{f}$ *with the same Lipschitz constant.*

Proof. Since, for every x, the interval $[0,x]$ is compact, by the extreme value theorem, the function $f(y)$ has a maximum on it, hence the value $\tilde{f}(x)$ exists and is strictly smaller than α. Therefore, $\tilde{f}$ is total on the interval $[0,\beta)$ and maps it in $[0,\alpha)$. By [10, Corollary 6.2.5], the function $\tilde{f}$ is computable on $[0,\beta)$. From $\tilde{x} \geq f(x)$ for every x and $\lim\limits_{x\nearrow\beta} f(x) = \alpha$, we obtain that $\lim\limits_{x\nearrow\beta} \tilde{f}(x) = \alpha$, which concludes the proof that $\tilde{f}$ is an $\mathbb{R}$-translation function from β to α.

In case f is Lipschitz continuous with a Lipschitz constant c, then we can proof the Lipschitz continuity of $\tilde{f}$ with the constant c by contradiction: supposing the existence of two reals $x_1 < x_2$ in $\mathrm{dom}(\tilde{f})$ such that

$$\tilde{f}(x_2) - \tilde{f}(x_1) > c(x_2 - x_1), \tag{8}$$

we fix two reals

$$y_1 \in [0,x_1] \text{ and } y_2 \in [0,x_2] \quad \text{such that} \quad \tilde{f}(x_1) = f(y_1) \text{ and } \tilde{f}(x_2) = f(y_2). \tag{9}$$

Then, by $\tilde{f}(x_1) < \tilde{f}(x_2)$ implies that y_2 does not lie on the interval $[0,x_1]$, hence we obtain that $y_2 \in (x_1,x_2]$. Therefore, it holds

$$0 < y_2 - x_1 \leq x_2 - x_1. \tag{10}$$

On the other hand, we obtain by choice of y_1 that $f(x_1) \leq f(y_1)$, hence

$$f(x_1) < f(y_1) = \tilde{f}(x_1) < \tilde{f}(x_2) - c(x_2 - x_1) = f(y_2) - c(x_2 - x_1) \leq f(y_2) - c(y_2 - x_1), \tag{11}$$

where the first inequality follows from (8) and the third one holds by the right part of (10).

Inequality (11) contradicts the Lipschitz continuity of f with the Lipschitz constant c for the points x_1 and y_2. □

Corollary 3. *If $\alpha \leq^{\mathbb{R}}_{\mathrm{S}} \beta$ for two reals α and β, then $\alpha \leq^{\mathbb{R}}_{\mathrm{S}} \beta$ via a nondecreasing $\mathbb{R}$-translation function.*

Similarly as for the monotone Solovay reducibility, we can show the closure upwards and downwards on left-c.e. reals relative to $\leq^{\mathbb{R}}_{\mathrm{S}}$.

Proposition 9. *Let $\alpha, \beta \in \mathbb{R}$ where α is left-c.e. Then, the following statements are equivalent:*

1. *β is left-c.e.;*
2. *There exists an $\mathbb{R}$-translation function from α to β;*
3. *There exists an $\mathbb{R}$-translation function from β to α.*

Proof. – If β is left-c.e., the translation functions function f from β to α and g from α to β can be defined explicitly by $f(x) = a_n + \frac{x-b_n}{b_{n+1}-b_n}(a_{n+1}-a_n)$ for every $x \in [b_n, b_{n+1})$ and $g(x) = b_n + \frac{x-a_n}{a_{n+1}-a_n}(b_{n+1}-b_n)$ for every $x \in [a_n, a_{n+1})$, respectively.

- If there exists an $\mathbb{R}$-translation function from α to β, then, by Proposition 8, we can fix a nondecreasing $\mathbb{R}$-translation function f from α to β. Then, given a left-c.e. approximation $a_0, a_1, \ldots$ of α, we compute a left-c.e. approximation of β step-wise: starting with $i_{-1} = -1$ and $m_{-1} = 0$, in every step $n \geq 0$, we approximate the values of $f(a_i)$ for all $i > i_{n-1}$ effectively until we find a natural $m_n > m_{n-1}$ and an index $j > i$ such that, for the approximated with tolerance 2^{-m_n} values $\tilde{f}^{(n)}(a_i)$ and $\tilde{f}^{(n)}(a_j)$ of $f(a_i)$ and $f(a_j)$, respectively, the inequality
$$\tilde{f}^{(n)}(a_j) + 2^{m_n} < \tilde{f}^{(n)}(a_i) - 2^{m_n}$$
holds, and set $b_n = \tilde{f}(a_i)$. Then the sequence $b_0, b_1, \ldots$ is a left-c.e. approximation of β.
- If there exists an $\mathbb{R}$-translation function from β to α, then, as in the previous case, we fix a nondecreasing $\mathbb{R}$-translation function f from β to α computed by a machine M. Then, given a left-c.e. approximation $a_0, a_1, \ldots$ of α, we compute a left-c.e. approximation of β step-wise: in every step $n \geq 0$, we search for a rational b_n such that there exists an index j_n such that $f(b_n) < a_{j_n}$ (this serves to ensure that $\max\{f(b) < \alpha\}$, and thus also $b_n < \beta$) and there exists a natural m_n such that the approximated with tolerance 2^{-m_n} value $\tilde{f}(b_n)$ of $f(b_n)$ satisfies the inequality
$$a_n + 2^{m_n} < \tilde{f}^{m_n}(b_n) - 2^{m_n} < \tilde{f}^{m_n}(b_n) + 2^{m_n} < a_{n+1} - 2^{m_n}.$$
Then the sequence $b_0, b_1, \ldots$ is a left-c.e. approximation of β.

□

Corollary 4. *The set of left-c.e. reals is closed downwards and upwards relative to* $\leq_{\mathrm{S}}^{\mathbb{R}}$.

In 2024, Kumabe, Miyabe, and Suzuki introduced [5, Chapter 5] a slightly modified version of $\leq_{\mathrm{S}}^{\mathbb{R}}$ by omitting the requirement for the $\mathbb{R}$-translation function f in Definition 6 to fulfill $f(x) < \alpha$ for all $x < \beta$. Kumabe et al. also examined the totalized variant $\leq_{\mathrm{cL}}^{\mathrm{loc}}$ of the latter reducibility.

Definition 7 (Kumabe et al., 2024). *A* WEAKLY $\mathbb{R}$-TRANSLATION FUNCTION *from a real* α *to another real* β *is a real function which is computable on the interval* $[0, \beta)$ *such that fulfills* (7).

A real α *is* COMPUTABLY LIPSCHITZ REDUCIBLE *to a real* β ON A C.E. OPEN INTERVAL, *written* $\alpha \leq_{\mathrm{cL}}^{\mathrm{open}} \beta$, *if there exists a Lipschitz continuous weakly* $\mathbb{R}$-*translation function* f *on reals from* β *to* α.

α *is* COMPUTABLY LIPSCHITZ REDUCIBLE *to a real* β LOCALLY, *written* $\alpha \leq_{\mathrm{cL}}^{\mathrm{loc}} \beta$, *if there exists a real* $b > \beta$ *and a Lipschitz continuous function* f *on reals, which is computable on* $[0, b]$ *and fulfills* $f(\beta) = \alpha$.

By [4, Theorem 1] and by [5, Observation 5.3], respectively, the reducibilities $\leq_{\mathrm{S}}^{\mathbb{R}}$ and $\leq_{\mathrm{cL}}^{\mathrm{open}}$ are equivalent to the Solovay reducibility $\leq_{\mathrm{S}}$ on the set of left-c.e. reals.

In Section 3, we prove that $\leq_{\mathrm{cL}}^{\mathrm{open}}$ is implied by $\leq_{\mathrm{S}}$ and $\leq_{\mathrm{cL}}^{\mathrm{loc}}$ is implied by $\leq_{\mathrm{S}}^{\mathrm{tot}}$ on $\mathbb{R}$ and that $\leq_{\mathrm{S}}^{\mathbb{R}}$ is implied by $\leq_{\mathrm{S}}$ on all but computable reals.

2 Reducibilities $\leq_{\mathbf{cL}}^{\mathbf{open}}$ and $\leq_{\mathbf{cL}}^{\mathbf{loc}}$ as measures of relative Martin-Löf and Schnorr randomness, respectively

On one hand, it is easy to see that every computable real α is $\leq_{\mathrm{cL}}^{\mathrm{loc}}$-reducible and thus also $\leq_{\mathrm{cL}}^{\mathrm{open}}$-reducible to every further real β via the weakly $\mathbb{R}$-translation function $f(x) = \alpha$ which is computable (as a real function) and totally defined on the unit interval.

On the other hand, if a real α is $\leq_{\mathrm{cL}}^{\mathrm{open}}$-reducible to a computable real β, then we can easily compute α as well; the formal proof is left to the reader as an exercise. The two latter observations imply that least $\leq_{\mathrm{cL}}^{\mathrm{open}}$- and $\leq_{\mathrm{cL}}^{\mathrm{loc}}$-degrees on $\mathbb{R}$ both contain exactly all computable reals.

Proposition 10. *The computable reals form the least degree on $\mathbb{R}$ relative to both $\leq_{\mathrm{cL}}^{\mathrm{open}}$ and $\leq_{\mathrm{cL}}^{\mathrm{loc}}$.*

The closure upwards of Martin-Löf random reals relative to Solovay reducibility has been proved by Solovay himself [7]; the closure upwards of Schnorr random reals relative to total Solovay reducibility has been demonstrated by Merkle and Titov [6, Corollary 2.10].

In what follows, we prove the same closures for the reducibilities $\leq_{\mathrm{cL}}^{\mathrm{open}}$ and $\leq_{\mathrm{cL}}^{\mathrm{open}}$, respectively.

Proposition 11. *1. The set of Martin-Löf random reals is closed upwards relative to $\leq_{\mathrm{cL}}^{\mathrm{open}}$.*
2. The set of Schnorr random reals is closed upwards relative to $\leq_{\mathrm{cL}}^{\mathrm{loc}}$.

Proof. 1. Let α and β, where β is Martin-Löf nonrandom, be two real that fulfill $\alpha \leq_{\mathrm{cL}}^{\mathrm{open}} \beta$ with a constant c via a real function f computed by a Turing machine M of type 2. In particular, for every rational $q < \beta$, f satisfies

$$|\alpha - f(q)\downarrow| < c(\beta - q). \tag{12}$$

We define further a two-argument-function $g : \mathbb{Q}|_{[0,1)} \times \mathbb{N} \to \mathbb{Q}$ by setting

$$g(q, m) = p_m \text{ where } M^{(q,q,\dots)} = (p_0, p_1, \dots), \text{ if defined.} \tag{13}$$

Then, for every rational $q < \beta$ and natural m, we know by the choice of M that $f(q)\downarrow - g(q, m) < 2^{-m}$, hence we obtain by (12) that

$$|\alpha - g(q, m)\downarrow| \leq c(\beta - q_m) + 2^{-m} \tag{14}$$

for every $q < \beta$ and $m \in \mathbb{N}$.

Let further $q_0, q_1, \dots$ be a standard enumeration of rationals on $[0, 1)$.

Since β is Martin-Löf nonrandom, we can fix further a Solovay test $S_0, S_1, \dots$ that fails on β, where we denote $S_n = [l_n, r_n]$ of length d_n for every n. In particular, this test has a finite measure:

$$M := \sum_{n\in\mathbb{N}} \mu(S_n) = \sum_{n\in\mathbb{N}} d_n < \infty. \tag{15}$$

Then we prove that α should be Martin-Löf nonrandom as well by constructing the following Solovay test: for every natural n, compute $g(l_n, n)$ and, if the computation halts, set

$$T_n = [g(l_n, n) - (cd_n + 2^{-n}), g(l_n, n) + (cd_n + 2^{-n})]. \quad (16)$$

Then, the test $T_0, T_1, \ldots$ that contains all defined T_n is computable and has a finite measure by

$$\sum_{n\in\mathbb{N}} \mu(T_n) \leq \sum_{n\in\mathbb{N}} (2^{-n+1} + 2d_n) = \sum_{n\in\mathbb{N}} 2^{-n+1} + \sum_{n\in\mathbb{N}} 2d_n = 4 + 2M < \infty. \quad (17)$$

Further, for every of infinitely many n such that $\beta \in S_n$, we have $l_n < \beta < r_n$, hence, by (14), we have $g(l_n, n)\downarrow$ (hence T_n is defined) and

$$|\alpha - g(l_n, n)| \leq c(\beta - l_n) + 2^{-n} < c(r_n - l_n) + 2^{-n} = cd_n + 2^{-n}, \quad (18)$$

which implies that $\alpha \in T_n$. Therefore, the test $T_0, T_1, \ldots$ that contains all defined T_n is a Solovay test that fails on α, hence α is Martin-Löf nonrandom.

2. If, in the latter proof, we f is totally defined on $[0, b]$ for some $b > \beta$ (i.e. $\alpha \leq^{\mathrm{loc}}_{\mathrm{cL}} \beta$ via f), then the function $\tilde{g}$ defined as in (13) fulfills (14) for every $q < b$ and $m \in \mathbb{N}$.
   If, additionally, $S_0, S_1, \ldots$ is a total Solovay test that fails on β (that witnesses the Schnorr nonrandomness of β by [1, Theorem 7.1.10]), then we can modify this test by replacing every S_n by $S_n \cap [0, b]$; the resulting test is, again, a total Solovay test that fails on β and has a computable measure:

$$M_{comp} := \sum_{n\in\mathbb{N}} \mu(S_n) = \sum_{n\in\mathbb{N}} d_n < \infty. \quad (19)$$

   For every natural n the interval T_n as in (16) is well-defined since we know from $l_n < b$ that $g(l_n, n)\downarrow$. Hence, the test $T_0, T_1, \ldots$ has a measure exactly equal to $4 + 2M_{comp}$, which is finite and computable. Moreover, for every of infinitely many n such that $\beta \in S_n$, it still holds by (18) that T_n is defined and contains α. Therefore, the test $T_0, T_1, \ldots$ is a total Solovay test that fails on α, hence α is Schnorr nonrandom.

□

3 Main result

Theorem 1. *For arbitrary reals α and β, the following implications hold:*

$$\alpha \leq_{\mathrm{S}} \beta \implies \alpha \leq^{\mathrm{open}}_{\mathrm{cL}} \beta \qquad \textit{and} \qquad \alpha \leq^{\mathrm{tot}}_{\mathrm{S}} \beta \implies \alpha \leq^{\mathrm{loc}}_{\mathrm{cL}} \beta. \quad (20)$$

Moreover, if α is not computable, then the following implication holds:

$$\alpha \leq_{\mathrm{S}} \beta \implies \alpha \leq^{\mathbb{R}}_{\mathrm{S}} \beta. \quad (21)$$

Proof. Let g be a $\mathbb{Q}$-translation function from β to α, and let α and β be two reals such that $\alpha \leq_{\mathrm{S}} \beta$ with some constant $c > \frac{1}{2^N}$ for some natural N via g. If β is left-c.e., then α is left-c.e. as well since the set LEFT−CE is closed downwards relative to $\leq_{\mathrm{S}}$, and the theorem statement follows by [4, Theorem 1]. So, in what follows, suppose that β is not left-c.e., i.e., $\mathrm{LC}(\beta)$ is not recursively enumerable.

Further, let b be the maximal real that fulfills the property

$$g(q)\downarrow \text{ for every } q < b. \tag{22}$$

It holds obviously that $b \geq \beta$ since g is a translation function from β to α, wherein, in case $\alpha \leq_{\mathrm{S}}^{\mathrm{tot}} \beta$, we obtain $b = 1$).

In the scope of the proof, we set an enumeration $q_0, q_1, \ldots$ of the domain of g. If there exists a rational $b > \beta$ such that there is no $q_i \in (\beta, b)$, then we obtain that $\mathrm{LC}(\beta) = \mathrm{dom}(g) \cap [0, \beta)$ is recursively enumerable, a contradiction. Therefore, there exists a subsequence of $q_0, q_1, \ldots$ which tends to β from above.

Construction of a weakly $\mathbb{R}$-translation function from $\boldsymbol{\beta}$ to $\boldsymbol{\alpha}$

Now, we construct a real weakly $\mathbb{R}$-translation function h from β to α in four steps: first, on the base of g we construct another partial function $\tilde{g}$ on rationals that witness the Solovay reducibility $\alpha \leq_{\mathrm{S}} \beta$ with the same constant c; second, on the base of $\tilde{g}$, we construct a technical computable two-argument partial function $f(\cdot, \cdot)$ on rationals; third, on the base of f we construct another technical two-argument partial function $\tilde{f}(\cdot, \cdot)$ on rationals; and fourth, on the base of $\tilde{f}$, we finally construct the function function $h(\cdot)$ on reals.

The lists of properties of functions $\tilde{g}$ through h are combined in Claims 1 — 4, respectively. The proofs of Claims 1 — 4 are given in the appendix.

First, we define a computable function $\tilde{g}$ from rationals as follows by setting

$$Q_n = \{q_m : m < n \text{ and } q_m \leq q_n\} \quad \text{and} \quad \tilde{g}(q_n) = \max\{g(q) : q \in Q_n\}. \tag{23}$$

Claim 1. *$\tilde{g}$ satisfies the following properties:*

$$\mathrm{dom}(\tilde{g}) = \mathrm{dom}(g), \tag{24}$$

$$0 < \alpha - \tilde{g}(q)\downarrow \leq \alpha - g(q) < c(\beta - q) \text{ for all } q \in \mathrm{LC}(\beta). \tag{25}$$

$$Q_m \subseteq Q_n \text{ and } \tilde{g}(q_m) \leq \tilde{g}(q_n) \text{ for such } q_m, q_n \in \mathrm{dom}(\tilde{g}) \text{ that } \begin{cases} m < n, \\ q_m < q_n. \end{cases} \tag{26}$$

From (24), we obtain that, in particular, $q_0, q_1, \ldots$ is at the same time a computable enumeration of $\mathrm{dom}(\tilde{g})$. The property (25) directly implies that the function g is a well-defined $\mathbb{Q}$-translation function that witnesses the Solovay reducibility of α to β with the same constant as g.

At the beginning of the second step, we fix some rational constant d and a natural $K > 1$ such that $2^K = d > c$ and define a computable two-argument partial function $f : \mathbb{Q} \times \mathbb{N} \to \mathbb{Q}$ as follows: given q and n, we enumerate $q_0(=0), q_1, \ldots$ and, in case if we meet after some enumeration step t a set $\{q_{i_0}(=0), q_{i_1}, \ldots, q_{i_k}\}$ where $i_0, i_1, \ldots, i_k \leq t$ such that

$$0 < q_{i_n} - q_{i_{n-1}} < 2^{-n} \text{ for all } i \in \{1, \ldots, k\} \text{ and} \tag{27}$$

$$0 \leq q - q_{i_k} < 2^{-n}, \tag{28}$$

let $\{q_{i_0}, q_{i_1}, \ldots, q_{i_N}\}$ be the greatest (under inclusion) such set of indexes not larger than t and define the ordered tuple

$$P(q,n) = (q_{i_0}, \ldots, q_{i_k}) \quad \text{and} \quad f(q,n) = \min\left\{\tilde{g}(p) + d(q-p) : p \in P(q,n)\right\}. \tag{29}$$

Accordingly, we write $P(q,n)\uparrow$ and $f(q,n)\uparrow$ if the search for the set $P(q,n)$ never ends.

Further, we define the (in case $b = \beta$ coinciding) sets

$$D_\beta = \{(q,n) : n \in \mathbb{N} \text{ and } q \in [0, \beta + 2^{-n})\} \text{ and} \tag{30}$$

$$D_b = \{(q,n) : n \in \mathbb{N} \text{ and } q \in [0, b + 2^{-n})\}, \tag{31}$$

that obviously fulfill $D_\beta \subseteq D_b$.

Claim 2. *f satisfies the following properties list of properties:*

$$P(q,n)\downarrow \text{ and } f(q,n)\downarrow \text{ for all naturals } n \text{ and rationals } q \in [0,b), \tag{32}$$

$$P(q,n)\downarrow \text{ and } f(q,n) \text{ for all } (q,n) \in D_b \tag{33}$$

$$f(q,n)\downarrow < \alpha + 2^{-(n-K-1)} \text{ for all } (q,n) \in D_\beta, \tag{34}$$

$$P(p,n)\downarrow \subseteq P(q,n) \text{ for all } (q,n) \in \operatorname{dom}(f) \text{ and } p < q, \tag{35}$$

$$\begin{cases} P(q,m)\downarrow \subseteq P(q,n) \\ 0 \leq f(q,m) - f(q,n)\downarrow < 2^{-(m-K)} \end{cases} \text{ for all } (q,n) \in \operatorname{dom}(f) \text{ and } m < n, \tag{36}$$

$$f(q,n) - f(p,n)\downarrow < d \cdot (q-p) \text{ for all } (q,n) \in \operatorname{dom}(f) \text{ and } p < q, \tag{37}$$

$$\alpha - f(q,n)\downarrow < c(\beta - q) \text{ for every } n \in \mathbb{N} \text{ and } q \in \operatorname{LC}(\beta). \tag{38}$$

In the second step, we define a computable two-argument partial function $\tilde{f} : \mathbb{Q} \times \mathbb{N} \to \mathbb{Q}$ by setting

$$\tilde{f}(q,n) = \max\left\{\{f(q',n) : q' \in P(q,n)\} \cup \{f(q,n)\}\right\} \tag{39}$$

for all $(q,n) \in \operatorname{dom}(f)$.

Claim 3. *$\tilde{f}$ satisfies the following list of properties:*

$$\mathrm{dom}(\tilde{f}) = \mathrm{dom}(f), \tag{40}$$

$$\tilde{f}(q,n) \geq f(q,n) \text{ for all } (q,n) \in \mathrm{dom}(\tilde{f}), \tag{41}$$

$$\tilde{f}(q,n) \downarrow < \alpha + 2^{-(n-K-1)} \text{ for all } (q,n) \in D_\beta, \tag{42}$$

$$|\tilde{f}(q,n) - \tilde{f}(q,m)| < 2^{-(m-K-1)} \text{ for all } (q,m) \in \mathrm{dom}(f) \text{ and } m < n, \tag{43}$$

$$|\tilde{f}(q,n) - \tilde{f}(p,n)\downarrow| < 2^{-(n-K)} + 2^K|q-p| \text{ for all } (q,n) \in \mathrm{dom}(f) \text{ and } p < q, \tag{44}$$

$$\alpha - \tilde{f}(q,n) \downarrow < c(\beta - q) \text{ for every } n \in \mathbb{N} \text{ and } q \in \mathrm{LC}(\beta). \tag{45}$$

In the third step, we define a real function

$$h(x) = \lim_{n\to\infty} f(q_n, n) \text{ for every effective approximation } q_n \underset{n\to\infty}{\to} x. \tag{46}$$

Claim 4. *h satisfies the following list of properties:*

$$[0,b) \in \mathrm{dom}(h); \tag{47}$$

$$h \text{ is Lipschitz continuous}; \tag{48}$$

$$h(x) \leq \alpha \text{ for every } x < \beta; \tag{49}$$

$$\lim_{\substack{x\to\beta \\ x\in[0,b)}} h(x) = \alpha; \tag{50}$$

$$h \text{ is (totally Type 2) computable on } (0,b). \tag{51}$$

The constructed weakly $\mathbb{R}$-translation function demonstrates the theorem statement

By (47) through (51), we obtain that h witnesses $\alpha \leq_{\mathrm{cL}}^{\mathrm{open}} \beta$ and, in case $b = 1$, even $\alpha \leq_{\mathrm{cL}}^{\mathrm{loc}} \beta$.

Now, in case α is not computable, we even obtain that $h(x) < \alpha$ for every $x < \beta$ by contradiction: assuming $h(\tilde{x}) = \alpha$ for some real $\tilde{x} < \beta$, we can fix two rationals $\tilde{q}$ and $\tilde{q}$ such that $\tilde{x} < \tilde{p} < \tilde{q} < \beta$. Then function $\tilde{h}$ defined on the compact interval $[0, \tilde{q}]$ by

$$\tilde{h}(x) = \max\{h(y) : y \in [0,x]\}$$

is computable on the whole interval $\tilde{p}, \tilde{q}$ by [10, Corollary 6.2.5]. On the other side, by (49), we have $h(x) \leq [0, \alpha]$ for every $x \in [0, \beta)$, hence our assumption implies that

$$\max\{h(y) : y \in [0,x]\} = h(\tilde{x}) = \alpha,$$

hence the function $\tilde{h}$ is a constant function defined on $\tilde{p}, \tilde{q}$ that returns α for every input. Hence α should be computable as a limit point of an effective approximation $g(\tilde{q}_0), g(\tilde{q}_1), \ldots$ where $q_0, q_1, \ldots$ is *any* computable effective approximation lying in $[\tilde{p}, \tilde{q}]$. A contradiction.

Thus, h is an $\mathbb{R}$-translation function from β to α, and we obtain from its Lipschitz continuity that $\alpha \leq_{\mathrm{S}}^{\mathbb{R}} \beta$.

□

On the other hand, by Proposition 4 that there exists a nonleft-c.e. real β such that the computable real 1 is Solovay reducible to it, while, By Corollary 4, the computable real 1 cannot be real Solovay reducible to β. These two observations imply together the following result.

Proposition 12. *There exists a computable real α and a nonleft-c.e. real β such that*

$$\alpha \leq_{\mathrm{S}} \beta \qquad \textit{and} \qquad \alpha \not\leq_{\mathrm{S}}^{\mathbb{R}} \beta.$$

The implication $\leq_{\mathrm{S}} \implies \leq_{\mathrm{cL}}^{\mathrm{open}}$ is strict on $\mathbb{R}$ since computable reals form the least degree in $\mathbb{R}$ relative to $\leq_{\mathrm{cL}}^{\mathrm{open}}$ (Proposition 10) but not relative to $\leq_{\mathrm{S}}$ ([6, Proposition 2.5]). We still don't know whether the implication $\leq_{\mathrm{S}} \implies \leq_{\mathrm{S}}^{\mathbb{R}}$ is strict on $\mathbb{R} \setminus \mathrm{COMP}$.

The implication $\leq_{\mathrm{S}}^{\mathrm{tot}} \implies \leq_{\mathrm{cL}}^{\mathrm{loc}}$ is strict since computable reals form the least degree in $\mathbb{R}$ (and thus also on $\mathbb{R}$) relative to $\leq_{\mathrm{cL}}^{\mathrm{loc}}$ (Proposition 10) but not relative to $\leq_{\mathrm{S}}^{\mathrm{tot}}$ ([6, Proposition 2.5]).

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A Appendix

Proof of Claim 1

We prove (24) through (26) step by step.

1. The equality (24) follows from the definition of $\tilde{g}$ since, for every n, we have $Q_n \in \mathrm{dom}(\tilde{g})$, hence $\tilde{g}(q_n)\downarrow$.
2. To prove (25), we first note that, for every $q < \beta$, there exists an index n such that $q = q_n$ since the enumeration of $\mathrm{dom}(g)$ contains $\mathrm{LC}(\beta)$. By construction, for all $q' \in Q_n$, we have $q' \leq q_n (< \beta)$, hence $0 \leq g(q') \downarrow < \alpha$. Therefore, we obtain that $0 < g(q_n) \downarrow < \alpha$. Further, the second inequality in (25) is implied by $g(q_n) \leq \tilde{g}(q)$, which, in turn, holds since q_n also lies in Q_n. The third inequality in (25) is straightforward since g witnesses the Solovay reducibility of α to β with the constant c.
3. If $m < n$ and $q_m < q_n < \beta$, then, for every $q_k \in Q_m$, it holds that $k \leq m < n$ and $q_k \leq q_m < q_n$, hence $q_k \in Q_n$. Therefore, we have $\tilde{g}(q_m) = \max\{g(q') : q' \in Q_m\} \leq \max\{g(q') : q' \in Q_n\} = \tilde{g}(q_n)$, which concludes the proof of (26).

Proof of Claim 2

We prove (32) through (38) step by step.

1. Let $n \in \mathbb{N}$ and $q \in [0, b)$. Then, for every rational $p \leq q$, it holds that $p < b$. By choice of b, the domain of $\tilde{g}$ contains all rationals on the interval $[0, b)$ and, by (24), we have $\mathrm{dom}(\tilde{g}) = \mathrm{dom}(g)$. Therefore, the value $\tilde{g}(p)$ is defined. Since the rational $p \in [0, q)$ has been chosen arbitrary, the search for $P(q, n)$ terminates, and thus $f(q, n)$ is defined.
2. Let $(q, n) \in D_b$. By $q < b + 2^{-n}$, there exists a rational $p \in [0, b)$ such that $q - p < 2^{-n}$. As we have already seen in the previous item, $q_0, q_1, \ldots$ contains all rationals on $[0, b)$, and thus also on $[0, p)$, hence the search for the set $P(q, n)$ will be terminated, and we obtain $f(q, n)\downarrow$.
3. Let $(q, n) \in D_\beta$. Since $D_\beta \subseteq D_b$, $P(q, n)$ and $f(q, n)$ are defined by (33). Next, let $p' \in P(q, n)$ be a by (27) and (28) existing element such that $q' \in (\beta - 2^{-n}, \beta)$ (the reader can easily check that $p' = q_{i_{k-1}}$ or $p' = q_{i_k}$). Then, it holds by (25) that

$$0 < \alpha - \tilde{g}(p') \quad \text{and} \quad q - p' < (\beta + 2^{-n}) - (\beta - 2^{-n}) = 2^{-(n-1)}. \tag{52}$$

Thus, we obtain that

$$f(q, n) \leq \tilde{g}(p') + d(q - p') \leq \alpha + d \cdot 2^{-(n-1)} = \alpha + 2^{-(n-K-1)},$$

where the first inequality holds by (29) since $p' \in P(q, n)$, and the second one is implied by (52).

4. Let $(q,n) \in \mathrm{dom}(f)$ (in particular, the set $P(q,n)$ is defined) and $p < q$. Fix $P(q,n) = (q_{i_0}, \dots, q_{i_k})$ and let N be the maximal index in the range $\{0,\dots,k\}$. such that $q_{i_N} < p$. Then, the set $\{q_{i_0}, \dots, q_{i_N}\}$ fulfills the inequalities (27) and (28) for q replaced by p. Therefore, $P(p,n) \downarrow\subseteq \{q_{i_0}, \dots, q_{i_N}\} \subseteq P(q,n)$.
5. Let $(q,n) \in \mathrm{dom}(f)$ and $m < n$. We first note that the set $P(q,n)$ obviously fulfills all inequalities in (27) and (28) for 2^{-n} replaced by 2^{-m}, hence the set $P(q,m)$ is defined and fulfills $P(q,m) \subseteq P(q,n)$; therefore, $f(q,n) \leq f(q,m)$.
   On the other hand, fix $q_x \in P(q,n)$ such that $f(q,n) = \tilde{g}(q_x) + d(q - p')$. Then, the inequalities (27) and (28) for the set $P(q,m)$ imply that there exists an element $q_y \in P(q,m)$ that fulfills
   $$q_y \leq q_x < q_y + 2^{-m} \qquad \text{and} \qquad x \geq y. \tag{53}$$
   In particular, the first and the third inequalities in the last line imply together by (26) if $y \neq x$ and directly if $x = y$ that
   $$\tilde{g}(q_y) \leq \tilde{g}(q_x). \tag{54}$$
   Therefore,
   $$f(q,m) \leq \tilde{g}(q_y) + d(q - q_y) \leq \tilde{g}(q_x) + d(q - q_x) + (q_x - q_y) < f(q,n) + 2^{-m},$$
   where the first equality holds since $y \in P(q,m)$, the second one is implied by (54), and the third one by choice of q_x, see the second inequality in (53).
6. Let $(q,n) \in \mathrm{dom}(f)$ and $p < q$. By (35), $P(p,n)$ is defined, and thus also $f(p,n)$, and we have
   $$P(p,n) \subseteq P(q,n). \tag{55}$$
   Further, we fix $p' \in P(p,n)$ such that $f(p,n) = \tilde{g}(p') + d(p - p')$. By (55), we also have $p' \in P(q,n)$, and thus
   $$f(q,n) \leq \tilde{g}(p') + d(q-p') \leq \tilde{g}(p') + d(p-p') + d(q-p) = f(p,n) + d(q-p).$$
7. Let $n \in \mathbb{N}$ and $q \in \mathrm{LC}(\beta)$. In particular, it holds $q < b$, hence $f(p,n)$ is defined by (32). We fix $p \in P(q,n)$ such that $f(q,n) = \tilde{g}(p) + (q - p)$. By $p \leq q$, we have
   $$\alpha - f(q,n) \downarrow= \alpha - \big(\tilde{g}(p) + (q-p)\big) \leq \alpha - \tilde{g}(p) < c(\beta - q),$$
   where the last inequality holds by (25).

Proof of Claim 3

We prove (40) through (45) step by step.

1. Let $(q,n) \in \mathrm{dom}(f)$. Then $(q,n) \in \mathrm{dom}(\tilde{f})$ because of the following argumentation: for every $q' \in P(q,n)$, it holds by definition of $P(q,n)$ that $q' < q$, hence, by (35), the set $P(q',n)$ is defined, and thus also $f(q',n)\downarrow$.

2. For every $(q,n) \in \mathrm{dom}(f)$, we have $(q,n) \in \mathrm{dom}(\tilde{f})$ by (40) and $\tilde{f}(q,n) \geq f(q,n)$ directly by definition of $\tilde{f}$.
3. Let $f(q,n) \in D_\beta$. Then, by (34), $f(q,n) \in \mathrm{dom}(f)$, hence, by (40), $f(q,n) \in \mathrm{dom}(\tilde{f})$. In case $\tilde{f}(q,n) = f(q,n)$, we obtain (42) by (36). Otherwise, there exists a rational $q' \in P(q,n)$ such that $\tilde{f}(q,n) = f(q',n)$. From $q' \in P(q,n)$, we obtain that $q' < q$, hence $(q',n) \in D_\beta$ by definition of D_β. Thus, by (34) again, we obtain $\tilde{f}(q,n) = f(q,n) < \alpha + 2^{-(n-K-1)}$.
4. Let $(q,n) \in \mathrm{dom}(f)$, and let $m < n$. Then we have $P(q,m) \subseteq P(q,n)$ by the first line of (36), hence, for every $p \in P(q,m)$, the value $f(p,m)$ is defined by the second line of (36) because the value $f(p,n)$ should be defined as well (since $q \in P(q,n)$). Therefore, the computation of $\tilde{f}(q,m)$ terminates.
Further, we will prove the inequality $\tilde{f}(q,m) - \tilde{f}(q,n) < 2^{-(m-K)}$.
In case $\tilde{f}(q,m) = f(q,m)$, we directly obtain

$$\tilde{f}(q,m) = f(q,m) < f(q,n) - 2^{-(n-K)} < \tilde{f}(q,n) - 2^{-(n-K)},$$

where the first inequality holds by the second line of (36), and the second inequality is implied by (41).
Otherwise, there exists $p \in P(q,m)$ such that $\tilde{f}(q,m) = f(p,m)$. Then we obtain by the first line of (36) that $p \in P(q,n)$, hence

$$\tilde{f}(q,n) \geq f(p,n) > f(p,m) - 2^{-(n-K)} = \tilde{f}(p,m) - 2^{-(n-K)},$$

where the second inequality holds by the second line of (36).
It remains to prove the inequality $\tilde{f}(q,n) - \tilde{f}(q,m) < 2^{-(m-K)}$.
In case $\tilde{f}(q,n) = f(q,n)$, we directly obtain that

$$\tilde{f}(q,n) = f(q,n) \leq f(q,m) \leq \tilde{f}(q,m) < \tilde{f}(q,m) - 2^{-(n-K)},$$

where the second inequality holds by the second line of (36).
Otherwise, there exists $p \in P(q,n)$ such that $\tilde{f}(q,n) = f(p,n)$. Note that, by the first line of (36), $f(p,m)\downarrow$. For $P(q,m) = (p_{i_0}, \ldots, p_{i_N})$, we fix an index M such that $p_{i_M} < p < p_{i_{M+1}}$. In particular, it means that

$$p_{i_M} < p < p_{i_M} + 2^{-m}. \tag{56}$$

Thus, by (37) applied for p_{i_M} and p (it is possible since $(p,m) \in \mathrm{dom}(f)$, as we have already seen), we obtain that

$$f(p,m) - f(p_{i_M}, m) < d(p - p_{i_M}) < d \cdot 2^{-m} < 2^{-(m-K).} \tag{57}$$

Therefore, it holds that

$$\tilde{f}(q,m) \geq f(p_{i_M}, m) > f(p,m) + 2^{-(m-K)} \geq f(p,n) + 2^{-(m-K)} = \tilde{f}(p,n) + 2^{-(m-K)},$$

where the first inequality holds since $p_{i_M} \in P(q,m)$, the second one is implied by (57), the third inequality holds by the second line of (36), and the equality holds by choice of p.

5. Let $(q, n) \in \mathrm{dom}(f)$ and $p < q$. By (37), it holds then that $(p, n) \in \mathrm{dom}(f)$, hence $P(p, n)$ and $\tilde{f}(p, n)$ are defined by (40).
Next, we will prove the inequality

$$\tilde{f}(p, n) - \tilde{f}(q, n) \leq d(q - p) + 2^{-(m-K)}. \tag{58}$$

In case $\tilde{f}(p, n) = f(p', n)$ for some $p' \in P(p, n)$, we obtain by the first line of (36) that $p' \in P(q, n)$, hence

$$\tilde{f}(q, n) \geq f(p', n) = \tilde{f}(p, n).$$

Otherwise, $\tilde{f}(p, n) = f(p, n)$. Then, for $P(p, n) = \{p_{i_0}, \ldots, p_{i_N}\}$, it holds in particular that

$$p_{i_N} \leq p < p_{i_N} + 2^{-n}. \tag{59}$$

Therefore, by (37) applied for p_{i_N} and p, we obtain that

$$f(p, n) - f(p_{i_N}, n) \leq d(p - p_{i_N}) < 2^{-n}. \tag{60}$$

By the first line of (36), we have $p_{i_N} \in P(q, n)$, hence we obtain the inequality

$$\tilde{f}(q, n) \geq f(p_{i_N}, n) > f(p, n) - 2^{-n} \geq \tilde{f}(p, n) - 2^{-n}, \tag{61}$$

which directly implies (58). Here, the second inequality holds by (60).
Further, we will prove the inequality

$$\tilde{f}(q, n) - \tilde{f}(p, n) \leq d(q - p) + 2^{-(m-K)}. \tag{62}$$

In case $\tilde{f}(q, n) = f(q, n)$, we immediately obtain that

$$\tilde{f}(q, n) = f(q, n) < f(p, n) + 2^{-(n-K)} \leq \tilde{f}(p, n) + 2^{-(n-K)},$$

where the first inequality holds by (37) since $p < q$, and the second one is implied by (41).
Otherwise, there exist $q' \in P(q, n)$ (hence $q' \leq q$) such that $\tilde{f}(q, n) = f(q', n)$. In particular, $(q', n) \in \mathrm{dom}(f)$. In case $q' \geq p$, we obtain by (36) applied for p and q' that $f(q', n) - f(p, n) < d(p' - q)$, hence

$$\tilde{f}(q.n) = f(q', n) \leq f(p, n) + d(p' - q) < \tilde{f}(p, n) + d(q' - p) \leq \tilde{f}(p, n) + d(q - q'),$$

where the second inequality holds by (41), and the third one by $q' \leq q$. In case $q' < p$, let $P(p, n) = \{p_{i_0}, \ldots, p_{i_N}\}$ and fix an index M such that $p_{i_M} \leq q' < p_{i_{M+1}}$. In particular, it means that

$$p_{i_M} \leq q' < p_{i_M} + 2^{-n}. \tag{63}$$

Thus, by (37) applied for p_{i_M} and q' (it is possible since $(q', n) \in \mathrm{dom}(f)$ as we have already seen), it holds that

$$f(q', n) - f(p_{i_M}, n) < d(p - p_{i_M}) < d \cdot 2^{-n} < 2^{-(n-K).} \tag{64}$$

So, we obtain the inequality

$$\tilde{f}(q,n) = f(q',n) < f(p_{i_M},n) + 2^{-(n-K)} \leq \tilde{f}(p,n) + 2^{-(n-K)},$$

which directly implies (62). Here, the equality holds by choice of p, the first inequality is implied by (64), and the second inequality holds since $p_{i_M} \in P(q,n)$. The inequality (58) and (62) together imply (44) since $d = \frac{1}{2^K}$.

6. Let $n \in \mathbb{N}$ and $q \in \mathrm{LC}(\beta)$. From (38), we know that $(q,n) \in \mathrm{dom}(f)$, hence, by (40), $(q,n) \in \mathrm{dom}(\tilde{f})$. Then (45) follows from

$$\alpha - \tilde{f}(q,n) \leq \alpha - f(q,n) < c(\beta - q),$$

where the first inequality holds by (41), and the second by (38).

Proof of Claim 4

We prove (47) through (51) step by step.

1. First, we prove (47). Fix a real $x \in [0,b)$ and an effective approximation $q_0, q_1, q_2, \dots$ of x. By Lemma 1(1), we obtain for every n that $x - q_n \leq 2^{-(n-1)}$, which implies that $q_n \leq x - 2^{-(n-1)} < b - 2^{-(n-1)}$. Therefore, $(q_n, n) \in D_b$ by definition of D_b, hence $f(q_n, n)\downarrow$ by (33). Next, for every $i \geq K+1$, it holds that

$$|\tilde{f}(q_{i+1}, i+1) - \tilde{f}(q_i, i)| \leq \underbrace{\underbrace{|\tilde{f}(q_{i+1}, i+1) - \tilde{f}(q_i, i+1)|}_{<2^{-((i+1)-K-1)} \text{ by } (43)} + \underbrace{|\tilde{f}(q_n, n+1) - \tilde{f}(q_n, n)|}_{<2^{-(i-K)} \text{ by } (44)}}_{<2^{-(i-1-K)}},$$

hence we obtain for every n that

$$|\tilde{f}(q_{(n+1)+K+2}, (n+1)+K+2) - \tilde{f}(q_{n+K+2}, n+K+2)| < 2^{-(n+1)}.$$

Thus, by definition, the sequence $(f(q_{K+2}, K+2), \tilde{f}(q_{K+3}, K+3), \dots)$ is an effective approximation. Denote its limit with y. To conclude that h is well-defined on $[0,b)$, it remains to prove that every further effective approximation $q'_0, q'_1, \dots$ of x converges to the same limit y.

Indeed, the sequence

$$\tilde{f}(q'_{K+2}, K+2) - \tilde{f}(q_{K+2}, K+2), \tilde{f}(q'_{K+3}, K+3) - \tilde{f}(q_{K+3}, K+3), \dots$$

converges to zero since, for every $n \in \mathbb{N}$, we have

$$\begin{aligned}
&|\tilde{f}(q'_{n+K+2}, n+K+2) - \tilde{f}(q_{n+K+2}, n+K+2)| \\
&= |\tilde{f}(q'_{n+K+2}, n+K+2) - y + y - \tilde{f}(q'_{n+K+2}, n+K+2)| \\
&\leq |y - \tilde{f}(q'_{n+K+2}, n+K+2)| + |y - \tilde{f}(q'_{n+K+2}, n+K+2)| \\
&< 2^{-(n+1)} + 2^{-(n+1)} = 2^{-n}.
\end{aligned}$$

where the latter inequality holds by Lemma 1(1) applied for the (by the previous discussion effective) approximations $(f(q_{K+2}, K+2), \tilde{f}(q_{K+3}, K+3), \dots)$ and $(f(q'_{K+2}, K+2), \tilde{f}(q'_{K+3}, K+3), \dots)$ of y.

2. To prove (48), we fix two reals x and x' such that $0 \leq x < x' < b$ and the effective approximations $q_0, q_1, \ldots$ and $q'_0, q'_1, \ldots$ of x and x', respectively. Then we obtain for every $n \in \mathbb{N}$ from (44) that

$$|\tilde{f}(q'_{n+K+2}, n+K+2) - \tilde{f}(q_{n+K+2}, n+K+2)| < 2^{-(n+2)} + 2^K |q'_{n+K+2} - q_{n+K+2}|.$$

Since we obviously have $h(x) = \lim_{i\to\infty} \tilde{f}(q_i, i) = \lim_{n\to\infty} \tilde{f}(q_{n+K+2}, n+K+2)$ and $h(x) = \lim_{i\to\infty} \tilde{f}(q'_i, i) = \lim_{n\to\infty} \tilde{f}(q'_{n+K+2}, n+K+2)$, the latter inequality implies in the limit that

$$|h(x') - h(x)| \leq \lim_{n\to\infty} \left(2^{-(n+2)} + 2^K |x' - x|\right) = 2^K |x' - x|.$$

3. Next, we prove (49). Fix a real $x \in [0, b)$ and an effective approximation $q_0, q_1, q_2, \ldots$ of x. By Lemma 1(1), we obtain for every n the inequality $x - q_n \leq 2^{-(n-1)}$, which implies that $q_n \leq x - 2^{-(n-1)} < \beta - 2^{-(n-1)}$. Therefore, $(q_n, n) \in D_\beta$ by definition of D_β. Then we obtain for every $n \in \mathbb{N}$ from (42) that

$$\tilde{f}(q_{n+K+2}, n+K+2) < \alpha + 2^{-(n+1)},$$

and thus, in the limit,

$$h(x) = \lim_{n\to\infty} (\tilde{f}(q_{n+K+2}, n+K+2)) \leq \alpha.$$

4. First, we note that, for obtaining (50), it suffice to show that

$$\lim_{x_n \nearrow \beta} h(x_n) = \alpha. \tag{65}$$

Indeed, in case $b = \beta$, we already have $\lim_{x_n \nearrow \beta} h(x_n) = \lim_{\substack{x_n \to \beta \\ x_n \in [0,b)}} h(x_n)$, while the case $b > \beta$ implies that $\beta \in [0, b)$, hence, by (48), the function h is Lipschitz continuous in β. In particular, it is continuous in β; thus, we have $\lim_{x_n \nearrow \beta} h(x_n) = h(\beta) = \lim_{\substack{x_n \to \beta \\ x_n \in [0,b)}} h(x_n)$.

In order to prove (65), we fix a strictly increasing sequence $x_0, x_1, \cdots \nearrow \beta$. First, we show that

$$\liminf_{n\to\infty} h(x_n) \leq \alpha. \tag{66}$$

Fix an index m and an effective approximation $q_0^m, q_0^m, \ldots$ of x_m. From $x_m < \beta$, we obtain for every $n \in \mathbb{N}$ that $(x_n^m, n) \in D_\beta$, hence, by (42), $\tilde{f}(q_n^m) < \alpha - 2^{-(n-K-1)}$. Letting n tend to infinity, the latter inequality turns to

$$h(x_m) = \lim_{n\to\infty} \tilde{f}(q_n^m, n) \leq \alpha.$$

For $m \to \infty$, the latter inequality yields (66).
Next, we show that

$$\liminf_{n\to\infty} h(x_n) \geq \alpha. \tag{67}$$

Fix an index m and a *strictly increasing* effective approximation $q_0^m, q_0^m, \dots$ of x_m. It particular, we have $q_0^m < q_1^m < \dots < x_m < \beta$, hence, for every $n \in \mathbb{N}$, we obtain from (45) that $\alpha - \tilde{f}(q_n^m) < c(\beta - q_n^m)$. Letting n tend to infinity, the latter inequality turns to

$$h(x_m) = \lim_{n\to\infty} \tilde{f}(q_n^m, n) \le c(\beta - x_m).$$

for $m \to \infty$, the latter inequality yields (67).
The inequalities (66) and (67) together imply (65).

5. In order to prove (51), we define a Turing machine M of Type 2 by setting

$$M^{(b_0,b_1,b_1,\dots)} = \big(\tilde{f}(b_{K+2}, K+2), \tilde{f}(b_{K+3}, K+3), \tilde{f}(b_{K+4}, K+4), \dots\big).$$

and show that M computes h on the interval $[0, b)$, which would directly imply that h is computable on $[0, b)$. We fix a real $x \in [0, b)$ and an effective approximation $x_0, x_1, \dots \to x$.
By Lemma 1(1), we obtain for every n that $x - x_n \le 2^{-(n-1)}$, which implies that $x_n \le x - 2^{-(n-1)} < b - 2^{-(n-1)}$. Therefore, $(x_n, n) \in D_b$ by definition of D_b, hence $f(x_n, n)\downarrow$ by (33). Next, for every $i \in \mathbb{N}$, it holds that

$$|\tilde{f}(b_{i+1}, i{+}1) - \tilde{f}(b_i, i)| \le \underbrace{\underbrace{|\tilde{f}(b_{i+1}, i+1) - \tilde{f}(b_i, i+1)|}_{<2^{-((i+1)-K-1)} \text{ by } (43)} + \underbrace{|\tilde{f}(b_n, n+1) - \tilde{f}(b_n, n)|}_{<2^{-(i-K)} \text{ by } (44)}}_{<2^{-(i-1-K)}},$$

hence we obtain for every n that

$$|\tilde{f}(b_{(n+1)+K+2}, (n+1)+K+2) - \tilde{f}(b_{n+K+2}, n+K+2)| < 2^{-(n+1)}.$$

Thus, by Lemma 1(2), the sequence $(f(b_{K+2}, K+2), \tilde{f}(b_{K+3}, K+3), \dots)$ is an effective approximation. This concludes the proof that M computes h on $[0, b)$ since we obviously have

$$\lim_{n\to\infty} f(b_{n+K+2}, n+K+2) = \lim_{n\to\infty} f(b_n, n) = h(n).$$