Total Solovay Reducibility and Uniform Schnorr Reducibility

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Notions of randomness

- A **Martin-Löf test** is a uniformly effective sequence of c.e. open sets $U_0, U_1, \ldots$, such that the open set U_n has uniform measure of at most 2^{-n-1}.
- A **Schnorr test** is a Martin-Löf test, such that the uniform measure of the open set U_n can be computed from n.

Definition

A sequence A is **Martin-Löf (Schnorr) random**, if there is no Martin-Löf (Schnorr) test $U_0, U_1, \ldots$, such that $A \in \bigcap_{i \in \omega} U_i$.

Further, we identify any real $\alpha := 0.A$ with its binary representation A.

Solovay reducibility

Definition

A real α is **Solovay reducible** to a real β, written $\alpha \leq_{\mathrm{S}} \beta$, if there is a constant $c > 0$ and a partial computable function $\varphi\colon \mathbb{Q} \to \mathbb{Q}$ such that for all $q < \beta$ it holds that $\varphi(q) \downarrow < \alpha$ and $\alpha - \varphi(q) < c(\beta - q)$.

- behaves badly outside from left-c.e. reals, so the Solovay-lattice on the field of left-c.e. reals will be considered only.
- The least upper bound of α and β is given by $\alpha + \beta$.
- Least degree: computable reals.
- Greastest degree: left-c.e. Martin-Löf random reals.

Schnorr reducibility

Definition

A real α is **Schnorr reducible** to a real β, written $\alpha \leq_{\mathrm{Sch}} \beta$, if there is a constant $c > 0$ such that for every computable measure machine B there exists a computable measure machine A so that

$$\mathrm{K_A}(\alpha \upharpoonright n) \leq \mathrm{K_B}(\beta \upharpoonright n) + c$$

A real α is **uniform Schnorr reducible** to a real β, written $\alpha \leq_{\mathrm{uSch}} \beta$, if there is a uniform way to transform B in A.

- Least degree: computable reals.
- Greatest degree: Schnorr random reals.
- There is a pair α, β of left-c.e. reals such that $\alpha \not\leq_{\mathrm{Sch}} \alpha + \beta$, so addition is not an upper bound for the Schnorr randomness (Miyabe, Nies and Stephan 2018).

Properties

The Schnorr random reals are not closed upwards in the Solovay degrees. (Miyabe, Nies and Stephan 2018)

Motivation

Is there a meaningful notion of reducibility, which implies both Solovay and Schnorr reducibilities?

Total Solovay reducibility

Definition

A real α is total Solovay reducible to a real β, written $\alpha \leq_{\mathrm{S}}^{\mathrm{tot}} \beta$, if there is a constant $c > 0$ and a total computable function $f\colon \mathbb{Q} \to \mathbb{Q}$ such that for all $q < \beta$ it holds that $f(q) < \alpha$ and $\alpha - f(q) < c(\beta - q)$.

Properties of total Solovay reducibility

- Obviously, the total Solovay reducibility implies the normal Solovay reducibility. The another implication is quite more interesting.

Theorem

For any left-c.e. reals α *and* β, $\alpha \leq_{\mathrm{S}}^{\mathrm{tot}} \beta$ *implies* $\alpha \leq_{\mathrm{uSch}} \beta$

The previous theorem implies the following straightforward corollaries:

- Martin-Löf random reals are closed upwards in the $\leq_S^{tot}$-degrees of left-c.e. reals.
- Schnorr random reals are closed upwards in the $\leq_S^{tot}$-degrees of left-c.e. reals.

No least degree

In contrast to the Solovay and Schnorr reducibilities, the computable reals are not total Solovay reducible to some special type of left-c.e. reals, namely the left-c.e. reals whose binary representation is a hyperimmune set.

Lemma

Let $\alpha = 0.A(0)\ldots$ *and* $\beta = 0.B(0)\ldots$ *be reals where the set* A *is computable. Then* α *is total Solovay reducible to* β *if and only if the set* B *is not hyperimmune.*

An infinite antichain

The last result can be extended to the following theorem:

Theorem

There exists a countably infinite antichain of $\leq_{S^2}$-incomparable left-c.e. reals, so that all of them are incomparable with the computable reals.

Modified version: an additional term

We can fix the non-reducibility of computable reals to something by weakening the Lipschitz condition.

Definition

A real α is **total Solovay-Rettinger-Zheng reducible** to a real β with respect to a total computable function $g : \mathbb{Q} \to \mathbb{Q}_{>0}$, written $\alpha \leq_{\mathrm{S}^*,\mathrm{g}} \beta$, if there is a constant c and a total computable function $f : \mathbb{Q} \to \mathbb{Q}$, such that if $q \in \mathbb{Q}$ and $q < \beta$, then $f(q) < \alpha$ and $\alpha - f(q) < c(\beta - q) + g(q)$.

Modified version: an additional term

If the additional term is bounded by $2^{-|q|}$ up to a positive factor from above, where $|q|$ means the length of the binary representation of a dyadic rational real q, then total Solovay-Rettinger-Zheng reducibility still implies uniform Schnorr reedućibility.

Theorem

For all left-c.e. α, β *and all* $d > 0$, $\alpha \leq_{\mathrm{S}^*, d2^{-|q|}} \beta$ *implies* $\alpha \leq_{\mathrm{uSch}} \beta$

Modified version: an additional term

On the other hand, if the additional term is computably bounded from below, then the total Solovay-Rettinger-Zheng reducibility has the same least degree as Solovay reducibility and Schnorr reducibility.

Theorem

For every computable α and every left-c.e. β, on holds for every positive-valued function g: if g dominates some monotone infinitely growing computable function, then $\alpha \leq_{\mathrm{S}^*,\mathrm{g}} \beta$

Total Baire-Solovay reducibility

Definition (total Baire-Solovay reducibility)

A real α is **total Baire-Solovay reducible** to a real β, written $\alpha \leq_{\mathrm{BS}}^{\mathrm{tot}} \beta$, if there is a constant $c > 0$ and a total computable functional $F\colon \omega^\omega \to \omega^\omega$, $(q_n)_{n\in\omega} \mapsto (r_n)_{n\in\omega}$, such that, if $q_n \nearrow \beta$ and $(q_n)_{n\in\omega}$ is strictly increasing, then $r_n \nearrow \alpha$ is strictly increasing and $\alpha - q_n < c(\beta - r_n)$ for every $n \in \omega$.

- Requiring $(q_n)_{n\in\omega}$ and $(r_n)_{n\in\omega}$ to be just non-decreasing is equivalent to total Solovay reducibility with the same constant c.
- Replacing F by a partial computable Baire functional is equivalent to Solovay reducibility with the same constant c.

Properties of total Baire-Solovay reducibility

- $\alpha \leq_{\mathrm{BS}}^{\mathrm{tot}} \beta \implies \alpha \leq_{\mathrm{S}} \beta$.
- Least degree: computable reals.

Lemma

The computable reals form the least degree of the Baire-Solovay lattice of left-c.e. reals.

- $\alpha \leq_{\mathrm{BS}}^{\mathrm{tot}} \beta \not\Rightarrow \alpha \leq_{\mathrm{S}}^{\mathrm{tot}} \beta$.

Open question: $\alpha \leq_{\mathrm{BS}}^{\mathrm{tot}} \implies \alpha \leq_{\mathrm{Sch}} \beta$?


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Колмогоровская сложность и алгоритмическая случайность
K(y, z)
K(x, z)
K(x, y)
2K(x, y, z) ⩽ K(x, y) + K(x, z) + K(y, z)