Solovay reducibility implies S2a-reducibility

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Abstract

The original notion of Solovay reducibility was introduced by Robert M. Solovay [3] in 1975 as a measure of relative randomness.

The S2a-reducibility introduced by Xizhong Zheng and Robert Rettinger [4] in 2004 is a modification of Solovay reducibility suitable for computably approximable (c.a.) reals.

We demonstrate that Solovay reducibility implies S2a-reducibility on the set of c.a. reals, even with the same constant, but not vice versa.

1 Introduction and background

We assume the reader to be familiar with the basic concepts and results of algorithmic randomness. Our notation is standard. Unexplained notation can be found in Downey and Hirschfeldt [2]. As it is standard in the field, all rational and real numbers are meant to be in the unit interval $[0, 1)$, unless stated otherwise.

We start by reviewing some central concepts and results that will be used subsequently.

Definition 1.1.

1. *A* COMPUTABLE APPROXIMATION *is a computable Cauchy sequence, i.e., a computable sequence of rational numbers that converges. A real is* COMPUTABLY APPROXIMABLE, *or* C.A., *if it is the limit of some computable approximation.*

2. *A* LEFT-C.E. APPROXIMATION *is a nondecreasing computable approximation. A real is* LEFT-C.E. *if it is the limit of some left-c.e. approximation.*

3. *A* RIGHT-C.E. APPROXIMATION *is a nonincreasing computable approximation. A real is* RIGHT-C.E. *if it is the limit of some right-c.e. approximation.*

In particular, if α is a left-c.e. real with a left-c.e. approximation $a_0, a_1, \dots$, then $1 - \alpha$ is right-c.e. real with a right-c.e. approximation $1 - a_0, 1 - a_1, \dots$.

Definition 1.2. *The* LEFT CUT *of a real α, written $LC(\alpha)$, is the set of all rationals strictly smaller than α.*

Definition 1.3 (Solovay [3], 1975). *A real α is* Solovay reducible *to a real β, written $\alpha \leq_S \beta$, if there exist a constant $c \in \mathbb{R}$ and a partially computable function g from the set $\mathbb{Q} \cap [0, 1)$ to itself such that, for all $q < \beta$, the value $g(q)$ is defined and fulfills the inequality*

$$0 < \alpha - g(q) < c(\beta - q). \tag{1}$$

We will refer to (1) as Solovay condition and to c as Solovay constant.

We say that the function g witnesses the Solovay reducibility of α to β in case g satisfies the conditions in Definition 1.3 for some Solovay constant. Note that such a function g is defined on the whole set $LC(\beta)$, maps it to $LC(\alpha)$, and fulfills

$$\lim_{q \nearrow \beta} g(q) = \alpha, \tag{2}$$

where $\lim_{q \nearrow \beta}$ denotes the left limit.

The set of left-c.e. reals is downwards closed under Solovay reducibility. This is usually shown using the index characterization of Solovay reducibility on the set of left-c.e. reals by Calude, Hertling, Khoussainov, and Wang [1], whereas the following proof uses the original definition.

Proposition 1.4. *If a real β is left-c.e., and a real α is Solovay reducible to β, then α is left-c.e.*

Proof. Let β be a left-c.e. real, and let $\alpha \leq_S \beta$ with Solovay constant c witnessed by some partial function g. Fix some left-c.e. approximation $b_0, b_1, \ldots$ of β, and define a computable sequence of rationals $a_0, a_1, \ldots$ by setting $a_n = g(b_n)$ for all n. By assumption on g, we obtain for every n the inequalities $a_n < \alpha$ and

$$0 < \alpha - a_n < c(\beta - b_n). \tag{3}$$

By $\lim_{n \to \infty} b_n = \beta$, these inequalities imply that $\lim_{n \to \infty} a_n = \alpha$, hence $a_0, a_1, \ldots$ is a computable approximation of α. Finally, setting $a'_n := \max\{a_m : m \leq n\}$ for every n provides a left-c.e approximation $a'_0, a'_1, \ldots$ of α, hence α is left-c.e. $\square$

Since the Solovay condition (1) is required only for rationals q in the left cut of β, many researchers focused on Solovay reducibility as a measure of relative randomness of left-c.e. reals, while, outside of the left-c.e. reals, the notion has been considered as "badly behaved" by several authors (see e.g. Downey and Hirschfeldt [2, Section 9.1]).

Zheng and Rettinger [4] introduced the reducibility $\leq_S^{2a}$ on the set of c.a. reals, which is equivalent to $\leq_S$ on the set of left-c.e. reals and, nowadays, is just called Solovay reducibility by some authors, even though it differs from Solovay reducibility as introduced in Definition 1.3 on the set of c.a. reals.

Definition 1.5 (Zheng, Rettinger [4], 2004). *A c.a. real α is S2a-*reducible *to a c.a. real β, written $\alpha \leq_S^{2a} \beta$, if there exist a constant $c \in \mathbb{R}$ and computable* *approximations $a_0, a_1, \dots$ and $b_0, b_1, \dots$ of α and β, respectively, that fulfill for every $n \in \mathbb{N}$ the inequality*

$$|\alpha - a_n| \leq c(|\beta - b_n| + 2^{-n}). \tag{4}$$

We will refer to (4) as S2a-CONDITION and to c as S2a-CONSTANT and say that the sequences $a_0, a_1, \dots$ and $b_0, b_1, \dots$ WITNESS the S2a-reducibility of α to β in case they satisfy the conditions of Definition 1.5 for some S2a-constant.

In the next section, we will show that on the set of c.a. reals, Solovay reducibility implies S2a-reducibility but not vice versa.

2 The theorem

Theorem 2.1. *Let α and β be reals where β is computably approximable and α is Solovay reducible to β with Solovay constant c. Then α is computably approximable as well, and α is S2a-reducible to β with S2a-constant c.*

Proof. Let the partially computable function g witness that α is Solovay reducible to β with Solovay constant c.

In case β is left-c.e., we directly obtain by Proposition 1.4 that α is left-c.e., as well, hence is computably approximable, and that thus we have $\alpha \leq_{\mathrm{S}}^{2\mathrm{a}} \beta$ with constant c since $\leq_{\mathrm{S}}$ and $\leq_{\mathrm{S}}^{2\mathrm{a}}$ coincide with preserving the constant c on the set of left-c.e. reals by Zheng and Rettinger [4, Theorem 3.2(2)]. So, in what follows, assume that β is not left-c.e. In particular, $\beta \neq 0$.

Let $b_0, b_1, \dots$ and $q_0, q_1, \dots$ be a computable approximation of β and a computable enumeration of $\mathrm{dom}(g)$, respectively, where, without loss of generality, we may assume that $b_0 = q_0 = 0$ since, by $\beta > 0$, $g(0)$ is defined.

Then, it suffices to inductively construct an infinite strictly increasing computable index sequence $i_0, i_1, \dots$ and a computable sequence $a_0, a_1, \dots$ of rationals that fulfill the inequality

$$|\alpha - a_n| < c(|\beta - b_{i_n}| + 2^{-n}) \tag{5}$$

for every n. Note that the sequence $b_0, b_1, \dots$ and its subsequence $b_{i_0}, b_{i_1}, \dots$ have the same limit β, hence by (5) the sequence $a_0, a_1, \dots$ converges to α.

Construction of the sequences At step 0, we set

$$i_0 = 0 \qquad \text{and} \qquad a_0 = g(q_{i_0}) = g(q_0) = g(0).$$

At step $n > 0$, we assume that i_{n-1} has already been defined. We say that a rational b SATISFIES REQUIREMENT R_n, if there exist a natural number $\ell \geq 2$ and indices $m_0, \dots, m_\ell$ such that the values $g(q_{m_0}), \dots, g(q_{m_\ell})$ are all defined and it holds that

$$b - 2^{-n-1} < q_{m_\ell} < b, \tag{6}$$

$$0 = q_{m_0} < \ldots < q_{m_\ell}, \tag{7}$$

$$q_{m_{k+1}} - q_{m_k} < 2^{-n-1} \quad \text{for all } k \in \{0, \ldots, \ell - 1\}, \tag{8}$$

$$0 < g(q_{m_\ell}) - g(q_{m_k}) < c(q_{m_\ell} - q_{m_k} + 2^{-n-2}) \quad \text{for all } k \in \{0, \ldots, \ell - 1\}. \tag{9}$$

At step n we search in parallel for $i > i_{n-1}$ such that $b = b_i$ satisfies requirement R_n, and for such i that is found first, we let $i_n = i$ and $a_n = g(q_{m_\ell})$ for the corresponding values of ℓ and m_ℓ.

Every construction step terminates Fix $n > 0$. We demonstrate that there is some $i > i_{n-1}$ such that b_i satisfies requirement R_n, hence step n will be completed successfully.

By definition of S2a-reducibility, the partial function g is defined on all rationals $q < \beta$, thus its domain is a dense subset of the real interval $[0, \beta]$, which has nonzero length because of $\beta \neq 0$. Therefore, we can fix a natural number $\ell \geq 2$ and indices $m_0, \ldots, m_{\ell-1}$ such that the values $g(q_{m_0}), \ldots, g(q_{m_{\ell-1}})$ are all defined, (7) and (8) hold with the conditions on m_ℓ removed, and we have

$$\beta - 2^{-n-2} < q_{m_{\ell-1}} < \beta. \tag{10}$$

Next, let

$$A := \max_{k \in \{0, \ldots, \ell-1\}} g(q_{m_k}). \tag{11}$$

By choice of g and (2), we have $A < \alpha$ and $\lim_{q \nearrow \beta} g(q) = \alpha$. Consequently, first, there exists a real $\varepsilon > 0$ such that

$$g(q) \in (A, \alpha) \quad \text{for all } q \in (\beta - \varepsilon, \beta).$$

Second, we can fix an index $m_\ell > \max\{m_0, \ldots, m_{\ell-1}\}$ such that

$$\max\{q_{m_{\ell-1}}, \beta - \varepsilon\} < q_{m_\ell} < \beta. \tag{12}$$

Then (7) and (8) hold, and the inequalities (10) and (12) imply

$$\beta - 2^{-n-2} < \beta - \varepsilon < q_{m_\ell} < \beta. \tag{13}$$

In order to show (9), fix $k \in \{0, \ldots, \ell - 1\}$. The first inequality in (9) holds because, by choice of ε and of m_ℓ, we have $g(q_{m_k}) < g(q_{m_\ell})$. The second one holds because we have

$$\begin{aligned} g(q_{m_\ell}) - g(q_{m_k}) &< \alpha - g(q_{m_k}) < c(\beta - q_{m_k}) \\ &= c(\beta - q_{m_\ell}) + c(q_{m_\ell} - q_{m_k}) < c(2^{-n-2} + q_{m_\ell} - q_{m_k}), \end{aligned}$$

where the first inequality holds by the left part of (1) for $q_{m_\ell} < \beta$, the second one holds by the right part of (1) for $q_{m_k} < \beta$, and the third one by (13).

Furthermore, (13) implies that all b that are close enough to β fulfill (6). Now the b_i converge to β, hence, for almost all i, the value $b = b_i$ fulfills (6). In particular, step n will complete successfully after finding $i > i_{n-1}$ as required.

The constructed sequences witness the S2a-reducibility In order to obtain $\alpha \leq_{\mathrm{S}}^{2\mathrm{a}} \beta$ with the S2a-constant c, it suffices to prove for every n the S2a-condition

$$|\alpha - a_n| < c(|\beta - b_{i_n}| + 2^{-n}). \tag{14}$$

So fix n. Let ℓ and $m_0, \dots, m_\ell$ be the witnesses found at step n for the fact that b_{i_n} satisfies requirement R_n. Thus, in particular, we have

$$a_n = g(q_{m_\ell}). \tag{15}$$

We prove (14) by distinguishing cases according to the position of β with respect to the values $0 = q_{m_0} < \cdots < q_{m_\ell}$.

- In case $q_{m_\ell} < \beta$, we have

  $$0 < \alpha - g(q_{m_\ell}) < c(\beta - q_{m_\ell}) \leq c(|\beta - b_{i_n}| + |b_{i_n} - q_{m_\ell}|) < c(|\beta - b_{i_n}| + 2^{-n-1}),$$

  where the first two inequalities follow from the Solovay condition, and the last one holds by (6).

- In case there exists $k \in \{0, \dots, \ell - 1\}$ such that $q_{m_k} < \beta \leq q_{m_{k+1}}$, on one hand, we have

  $$\alpha - g(q_{m_\ell}) \leq \alpha - g(q_{m_k}) < c(\beta - q_{m_k}) < c \cdot 2^{-n-1} < (|\beta - b_{i_n}| + 2^{-n}),$$

  where the first three inequalities follow, from left to right, by the lower bound in (9), by the Solovay condition, and, finally, because β is contained in the interval $(q_{m_k}, q_{m_{k+1}}]$, which has length strictly less than 2^{-n-1}.

  On the other hand, we also obtain an upper bound for $g(q_{m_\ell}) - \alpha$:

  $$\begin{aligned} g(q_{m_\ell}) - \alpha &< g\big(q_{m_\ell}\big) - g(q_{m_k}\big) \\ &\leq c\big(q_{m_\ell} - q_{m_k} + 2^{-n-1}\big) \\ &\leq c\big(q_{m_\ell} - \beta + 2^{-n-1} + 2^{-n-1}\big) \\ &< c\big(b_{i_n} - \beta + 2^{-n}\big) \leq c(|b_{i_n} - \beta| + 2^{-n}), \end{aligned}$$

  where the first four inequalities follow, from top to bottom, (i) by $q_{m_k} < \beta$ and the choice of g, (ii) by (9), (iii) because, by case assumption, β differs from q_{m_k} by at most 2^{-n-1}, and (iv) by $q_{m_\ell} < b_{i_n}$.

  The upper bounds for $\alpha - g(q_{m_\ell})$ and $g(q_{m_\ell}) - \alpha$ imply together that

  $$|\alpha - g(q_{m_\ell})| < c(|\beta - b_{i_n}| + 2^{-n}).$$

The cases above are exhaustive with respect to the possible positions of β, hence, by $a_n = g(q_{m_\ell})$, inequality (14) follows. This completes the proof of Theorem 2.1. □

We conclude by observing that the converse of Theorem 2.1 fails, i.e., that there are c.a. reals α and β such that $\beta \leq_{\mathrm{S}}^{2\mathrm{a}} \alpha$ but $\beta \not\leq_{\mathrm{S}} \alpha$. In fact, α can be chosen to be left-c.e.

Proposition 2.2. *Let α be a real that is left-c.e. but not right-c.e. Then it holds that*

$$1-\alpha \leq_{\mathrm{S}}^{2\mathrm{a}} \alpha \qquad \text{and} \qquad 1-\alpha \not\leq_{\mathrm{S}} \alpha.$$

Proof. The real α is not right-c.e., hence $1-\alpha$ is not left-c.e. This implies that $1-\alpha \not\leq_{\mathrm{S}} \alpha$ since the set of left-c.e. reals is closed downwards under Solovay reducibility by Proposition 1.4.

On the other hand, for a left-c.e. approximation $a_0, a_1, \dots$ of α, the sequence $1-a_0, 1-a_1, \dots$ is a right-c.e. approximation of $1-\alpha$. The computable approximations $1-a_0, 1-a_1, \dots$ and $a_0, a_1, \dots$ obviously fulfill the S2a-condition (4) with constant $c=1$, hence $1-\alpha \leq_{\mathrm{S}}^{2\mathrm{a}} \alpha$. □

References

[1] Christian Calude, Peter Hertling, Bakhadyr Khoussainov, and Yongge Wang. Recursively enumerable reals and Chaitin Ω numbers. In M. Morvan, C. Meinel, and D. Krob, editors, *STACS 98. 15th Annual Symposium on Theoretical Aspects of Computer Science 1998. Proceedings*, volume 1373 of Lecture Notes in Computer Science, pp 596–606. Springer, Berlin (1998)

[2] Rodney Downey and Denis Hirschfeldt. *Algorithmic Randomness and Complexity.* Springer, Berlin (2010)

[3] Robert Solovay. Draft of paper (or series of papers) on Chaitin's work. *Unpublished notes*, 215 pages (1975)

[4] Xizhong Zheng and Robert Rettinger. On the extensions of Solovay-reducibility. *COCOON 2004: Computing and Combinatorics*, pp 360–369 (2004)